LECTURE 2: ROW REDUCTION AND ECHELON FORMS

Tuesday, September 17, 2019 4:34 PM

Welcome to the second part of our systems of equations extravaganza!
Today we'll do more examples, and I'll also give you a more systematic way of solving systems of equations

I- ROW-ECHELON FORM

Example: Solve the following system

$$\begin{cases} x + 3y + 5z = 6 \\ x - 2y + 4z = -8 \\ \quad\; y + 3z = 0 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 5 & 6 \\ 1 & -2 & 4 & -8 \\ 0 & 1 & 3 & 0 \end{array}\right] (\times -1) \longrightarrow \left[\begin{array}{ccc|c} 1 & 3 & 5 & 6 \\ 0 & -5 & -1 & -14 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

WANT = 0

(1 s on top)

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 3 & 5 & 6 \\ 0 & 1 & 3 & 0 \\ 0 & -5 & -1 & -14 \end{array}\right] (\times 5)$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 3 & 5 & 6 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 14 & -14 \end{array}\right]$$

~~TRIANGULAR~~ FORM

ROW-ECHELON FORM

Note: Pivots are 1, 1, 14 (<u>always</u> nonzero)

> **Definition:** ROW-ECHELON FORM (REF)
> 1) Entries below and to the left of a pivot are nonzero
> 2) Rows of 0's are at the bottom

Non Examples:

$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 4 & 5 & 6 & 7 \\ 0 & 9 & 10 & 11 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 4 & 5 \\ 0 & 6 & 7 & 8 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 4 & 5 \end{bmatrix}$$

<u>BACK TO:</u>

$$\left[\begin{array}{ccc|c} 1 & 3 & 5 & 6 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 14 & -14 \end{array}\right]$$

Could Backsubstitute, but better: Continue row reducing

WANT $= 0$

$$\to \left[\begin{array}{ccc|c} 1 & 3 & 5 & 6 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right] (\times -3)$$

$$\to \left[\begin{array}{ccc|c} 1 & 3 & 5 & 6 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right] (\times -5)$$

$$\to \left[\begin{array}{ccc|c} 1 & 3 & 0 & 11 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right] (\times -3)$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

REDUCED ROW-ECHELON FORM

Solution: $\boxed{\begin{cases} x = 2 \\ y = 3 \\ z = -1 \end{cases}}$

NEAT-O!

(No Backsubstitution Required)

Definition: REDUCED ROW-ECHELON FORM

1) Pivots = 1
2) All the other entries in the column are 0

Non-Examples

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

NEAT FACT: RREF is unique

II- INFINITELY MANY SOLUTIONS

Here is why RREF is so useful

Example: Solve

$$\begin{cases} x + 3y + z + t = 1 \\ -4x - 9y + 2z - t = -1 \\ -3y - 6z - 3t = -3 \\ y + 2z + t = 1 \end{cases}$$

$$(\div -3)\left[\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 1 \\ -4 & -9 & 2 & -1 & -1 \\ 0 & -3 & -6 & -3 & -3 \\ 0 & 1 & 2 & 1 & 1 \end{array}\right] (\times 4) \to \left[\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 1 \\ 0 & 3 & 6 & 3 & 3 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 1 & 1 \end{array}\right] (\div 3)$$

$$\to \left[\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 1 & 1 \end{array}\right] (\times -1)\ (\times -1)$$

Don't eliminate ALL the rows!

WANT = 0

$$\to \left[\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] (\times -3)$$

ROW-ECHELON FORM

$$\to \begin{array}{c} \\ \left[\begin{array}{cccc|c} 1 & 0 & -5 & -2 & -2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ \qquad\quad z \quad\ t \end{array}$$

REDUCED ROW-ECHELON FORM

Backsubstitution

$$\begin{cases} x - 5z - 2t = -2 \\ y + 2z + t = 1 \\ 0 = 0 \\ 0 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x = -2 + 5z + 2t \\ y = 1 - 2z - t \end{cases}$$ $\rightsquigarrow$ True for every z & t !

General Tso'lution:

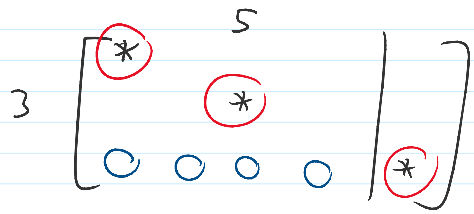
$$\begin{cases} x = -2 + 5z + 2t \\ y = 1 - 2z - t \\ z = z \\ t = t \end{cases}$$ FREE VARIABLES

Note: The free variables are ALWAYS in the non-pivot columns
(Here: 3rd and 4th columns)

III- PIVOTS

UPSHOT: Pivots are IMPORTANT! They tell us if a system has a solution or not!

Example: Suppose a system has a 3 x 5 augmented matrix whose last column is a pivot column.
Is the system consistent?

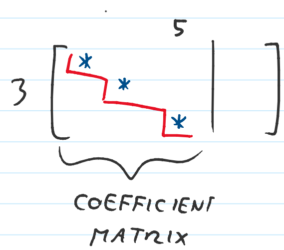


KNOW:

1) Pivot in the 5th row
2) Pivots are nonzero
3) Entries to the left of a pivot are 0 (by REF)

We have a row of the form [0 0 0 0 | BLAH] with BLAH $\neq 0$
Hence the system is inconsistent by the important fact from the previous lecture

Example: (if time permits)
Same, but this time pivot in every row of the coefficient matrix



No row of the form [0 0 0 0 BLAH]

$\Rightarrow$ $\boxed{\text{CONSISTENT}}$