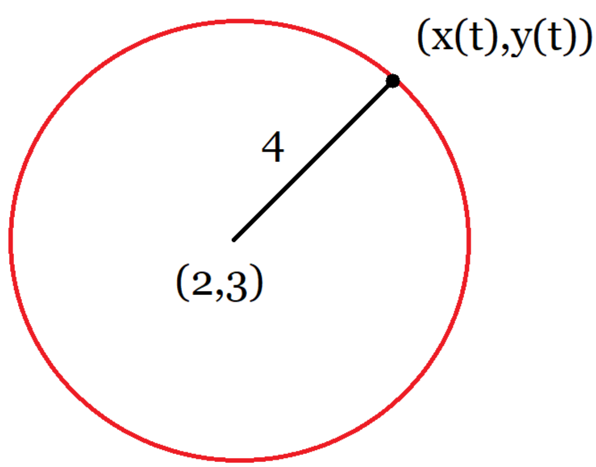
LECTURE 10: LINE INTEGRALS (I)

1. Review: Parametric Equations

Video: Parametric Equations

Today: Line Integrals, which essentially boils down to parametric equations, so let's first review the 3 most important parametrizations.

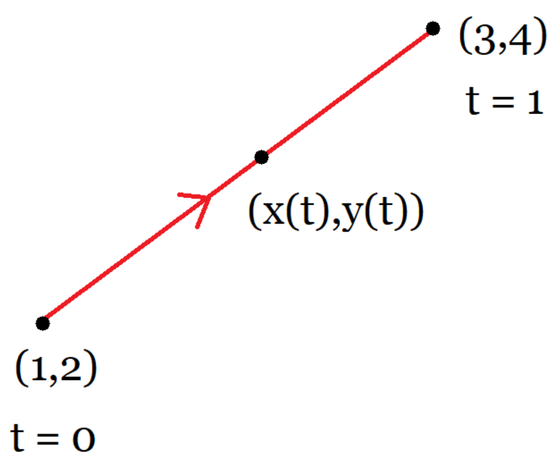
Example 1: Find parametric equations for the circle centered at $(2, 3)$ and radius 4



Date: Monday, January 27, 2020.

$$
\begin{aligned}
x(t) =& 2 + 4\cos(t) \\
y(t) =& 3 + 4\sin(t) \\
& (0 \leq t \leq 2\pi)
\end{aligned}
$$

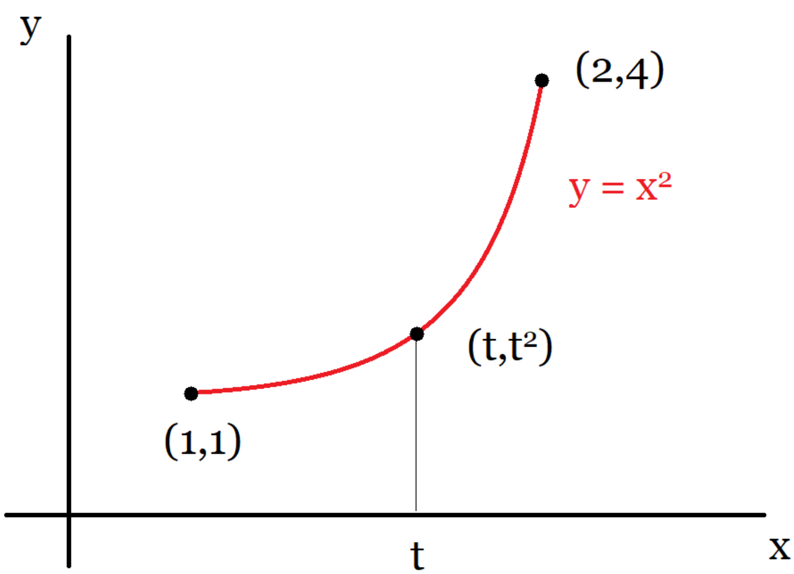
Example 2: Same but the line segment connecting $(1, 2)$ and $(3, 4)$



$$
\begin{aligned}
x(t) =& (1-t)\,1 + t3 = 1 + 2t \\
y(t) =& (1-t)\,2 + t4 = 2 + 2t \\
& (0 \leq t \leq 1)
\end{aligned}
$$

Note: In case you're having doubts about whether to put t or $1-t$, simply plug in $t = 0$, in your equations, which should give you $(1, 2)$

Example 3: Same but the parabola $y = x^2$ connecting $(1, 1)$ and $(2, 4)$



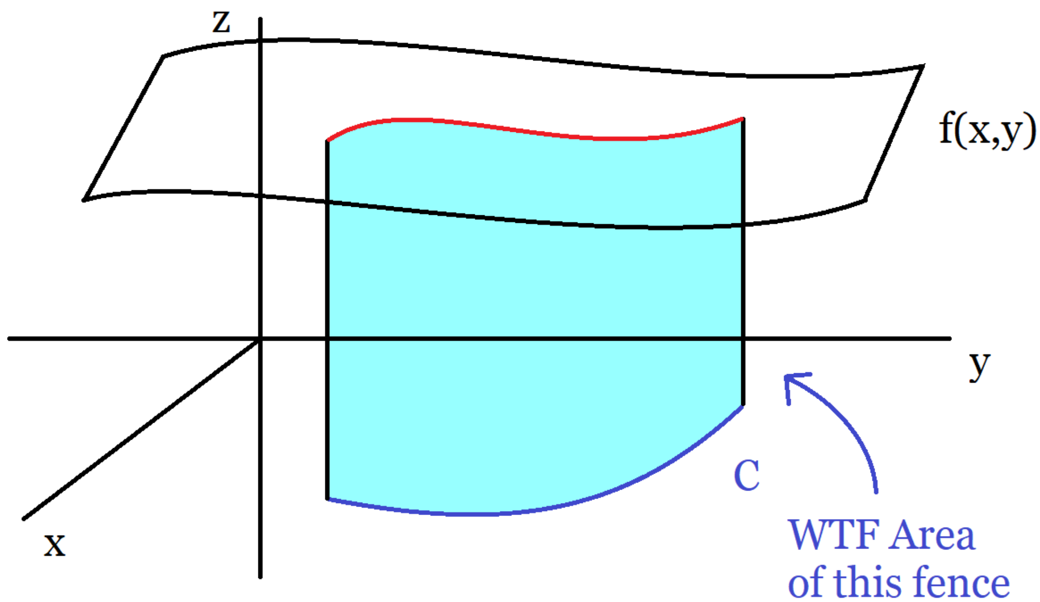
$$x(t) = t$$
$$y(t) = t^2 \quad (1 \leq t \leq 2)$$

2. Line Integrals

Video: Line Integral

Really cool! In calculus, you integrated a function f over an interval $[a, b]$ but today we'll integrate a function over any curve!

Goal: Given a curve C and a function $f(x, y)$, find the area of the fence under f and over C



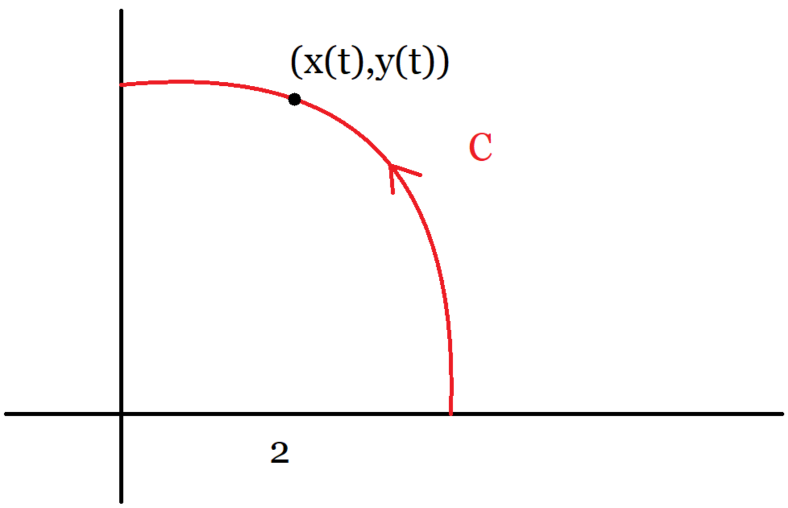
Notation: $\int_C f(x,y)ds$ = Area under f and over C

Example:

$$\int_C x^2 y \, ds$$

C : Quarter circle $x^2 + y^2 = 4$ in the first quadrant (counterclockwise)

(1) **Picture:**



(2) **Parametrize C:**

$$\begin{aligned} x(t) &= 2\cos(t) \\ y(t) &= 2\sin(t) \quad \left(0 \leq t \leq \frac{\pi}{2}\right) \end{aligned}$$

(3) **Integrate:** Use following fact (will derive it later, but for now think of it as a Jacobian):

$$ds = \sqrt{(x'(t))^2 + (y'(t))^2} dt$$

$$\begin{aligned}\int_C x^2 y\, ds &= \int_0^{\frac{\pi}{2}} (x(t))^2 (y(t)) \sqrt{(x'(t))^2 + (y'(t))^2}\, dt \\ &= \int_0^{\frac{\pi}{2}} (2\cos(t))^2\, 2\sin(t) \sqrt{(-2\sin(t))^2 + (2\cos(t))^2} dt \\ &= \int_0^{\frac{\pi}{2}} 8\cos^2(t)\sin(t)\sqrt{4\sin^2(t) + 4\cos^2(t)} dt \\ &= \int_0^{\frac{\pi}{2}} 8\cos^2(t)\sin(t)\sqrt{4} dt \\ &= 16\left[-\frac{1}{3}\cos^3(t)\right]_0^{\frac{\pi}{2}} \qquad (u = \cos(t)) \\ &= \frac{16}{3}\end{aligned}$$

Remarks:

(1) Small mnemonic: $ds = \sqrt{(dx)^2 + (dy)^2}$ (diagonal length)

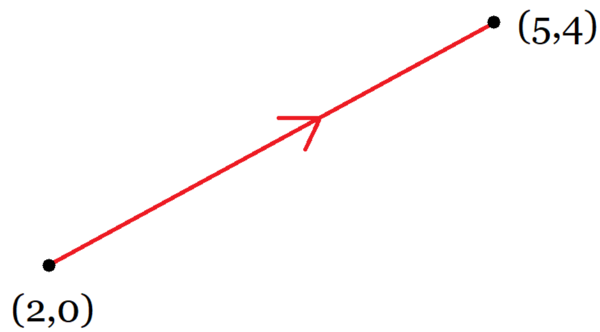
(2) Compare with Length$(C) = \int \sqrt{(x'(t))^2 + (y'(t))^2} dt$ (from Math 2D), so it's like a length, but with height f

Example:
Video: Line Integral Example

$$\int_C y ds$$

C : Line from $(2, 0)$ to $(5, 4)$

(1) **Picture:**



(2) **Parametrize:**

$$
\begin{aligned}
x(t) = &(1-t)\,2 + t\,(5) = \ 3t+2 \\
y(t) = &(1-t)\,0 + t\,(4) = \ 4t \qquad\quad (0 \leq t \leq 1)
\end{aligned}
$$

(3) **Integrate:**

$$\begin{aligned}
\int_C y\,ds &= \int_0^1 y(t)\sqrt{(x'(t))^2+(y'(t))^2}\,dt \\
&= \int_0^1 4t\sqrt{3^2+4^2}\,dt \\
&= \int_0^1 4t(5) \\
&= \int_0^1 20t \\
&= \cdots \\
&= 10
\end{aligned}$$

Interpretations

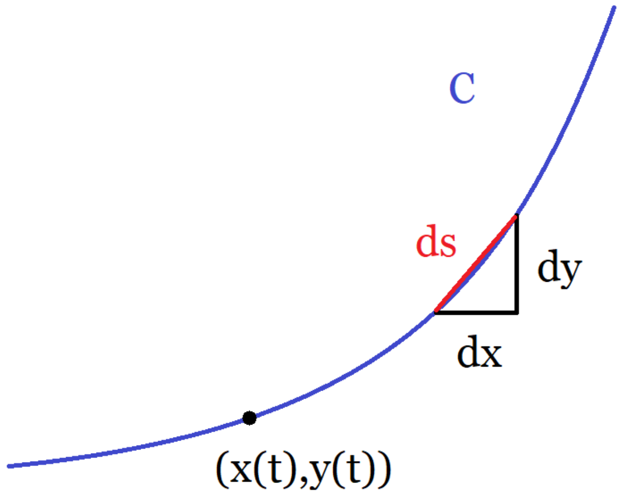
- $\int_C f =$ Area of fence under f, over C
- If $f =$ Force, $\int_C f\,ds =$ Work done by f over C
- If $f =$ Density, then $\int_C f\,ds =$ Mass of (wire) C

Note: For even more practice, you can check out this older video: Line Integral of a Function

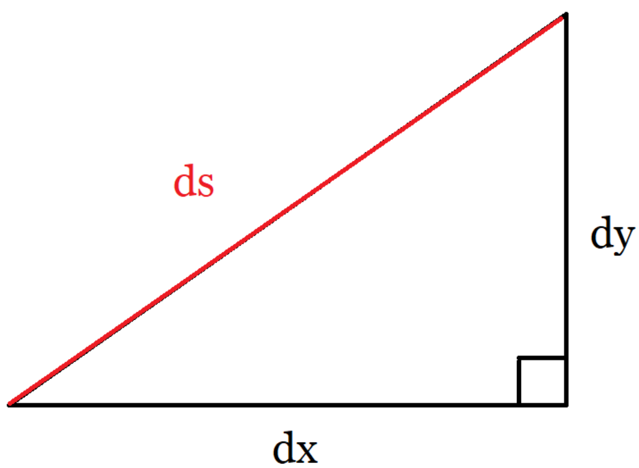
3. What is ds ?

Video: Line Integral Derivation

Idea: If C is a curve and you change x and y a little bit to get dx and dy, then ds is just the change in diagonal (which is close to C)



To find ds, just use the Pythagorean Theorem



$$ds = \sqrt{(dx)^2 + (dy)^2}$$

Important Trick:

$$\begin{aligned} dx &= \frac{dx}{dt}dt = x'(t)dt \\ dy &= \frac{dy}{dt}dt = y'(t)dt \end{aligned}$$

So ds is just

$$\begin{aligned} ds = \sqrt{(dx)^2 + (dy)^2} &= \sqrt{(x'(t)dt)^2 + (y'(t)dt)^2} \\ &= \sqrt{(x'(t))^2(dt)^2 + (y'(t))^2(dt)^2} \\ &= \sqrt{(x'(t))^2 + (y'(t))^2}\, dt \end{aligned}$$

Finally, to get $\int_C f$, just multiply the above by f and integrate:

$$\int_C f(x,y)ds = \int_a^b f(x(t), y(t))\sqrt{(x'(t))^2 + (y'(t))^2}dt$$

4. 3 dimensions

Video: Integral over a helix

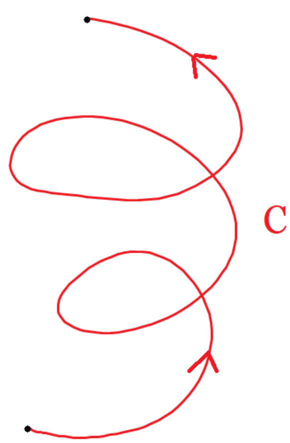
And of course, everything here can be generalized to 3 dimensions:
Example:

$$\int_C x^2 + y^2 + z^2 \, ds$$

C : Helix parametrized

$$\begin{aligned} x(t) &= \cos(t) \\ y(t) &= \sin(t) \\ z(t) &= t \\ (0 \leq t \leq 6\pi) \end{aligned}$$

(1) **Picture:**



(2) **Parametrize:** ✓

(3) **Integrate:**

$$\begin{aligned}
\int_C x^2 + y^2 + z^2 \, ds &= \int_0^{6\pi} \left((x(t))^2 + (y(t))^2 + (z(t))^2\right) \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} \\
&= \int_0^{6\pi} \left(\cos^2(t) + \sin^2(t) + t^2\right) \sqrt{(-\sin(t))^2 + (\cos(t))^2 + 1^2} dt \\
&= \int_0^{6\pi} \left(1 + t^2\right) \sqrt{2} dt \\
&= \sqrt{2}\left(6\pi + \frac{(6\pi)^3}{3}\right)
\end{aligned}$$