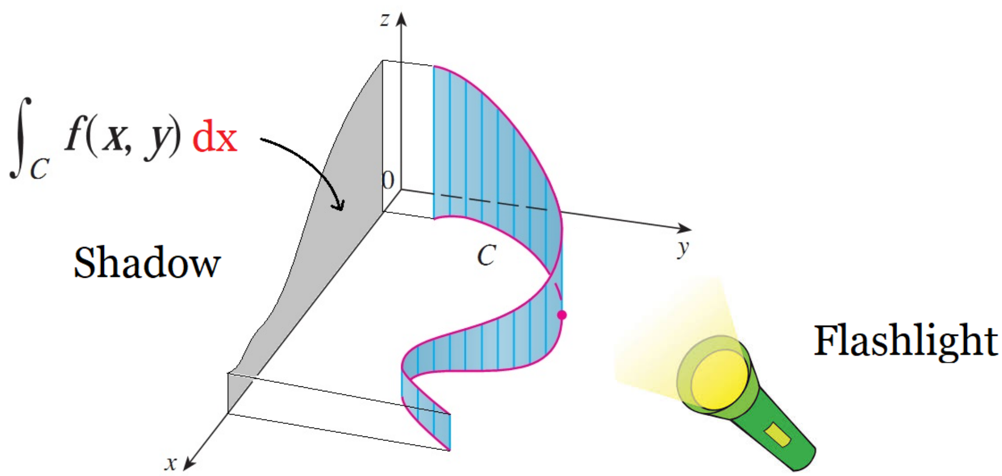
LECTURE 11: LINE INTEGRALS (II)

Last time: We discovered the line integral, which calculates the area of the fence under f and over C.

1. Two Variations

Video Line integral with respect to x

Two variations on this idea: Suppose you have a flashlight in the y-direction, and you cast a light on the fence, then you get a shadow on the left [1]



Date: Friday, January 30, 2020.

[1] Source: Math Stackexchange

$$\begin{aligned}\text{Area of Shadow on the left } &= \int_C f(x,y)\color{red}{dx} \\ &= \int_C f(x,y)\frac{dx}{\color{red}{dt}}\color{red}{dt} \\ &= \int_a^b f(x(t),y(t))x'(t)dt\end{aligned}$$

Similarly, if you have a flashlight in the $x-$direction:

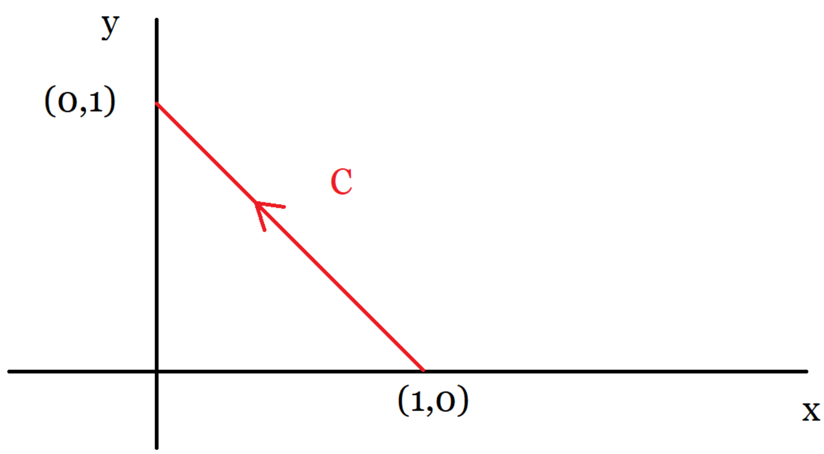
$$\begin{aligned}\text{Area of Shadow in the back } &= \int_C f(x,y)\color{red}{dy} \\ &= \int_C f(x,y)\frac{dy}{\color{red}{dt}}\color{red}{dt} \\ &= \int_a^b f(x(t),y(t))y'(t)dt\end{aligned}$$

Can combine the two:

Example: Calculate $\int_C -ydx + xdy$

(a) C is the line segment from $(1,0)$ to $(0,1)$

(1) **Picture**



(2) **Parametrize**

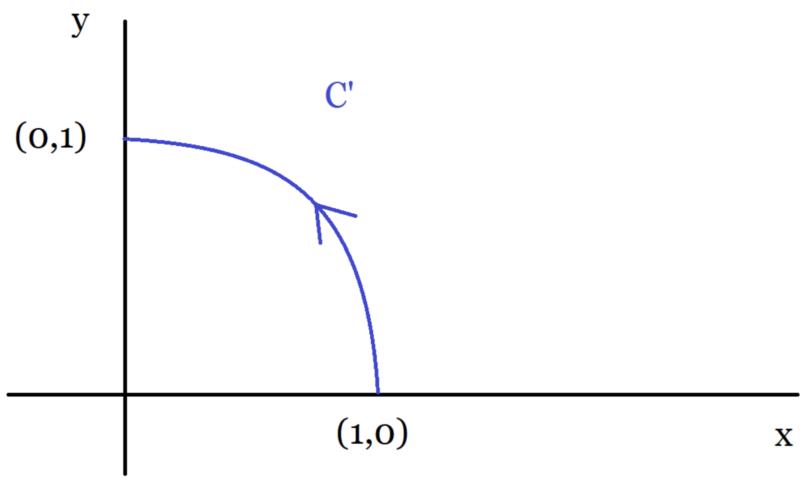
$$\begin{aligned} x(t) =& (1-t)\,1 + t\,0 = 1-t \\ y(t) =& (1-t)\,0 + t\,1 = t \\ (0 \leq t \leq 1) \end{aligned}$$

(3) **Integrate**

$$\begin{aligned} \int_C -y\frac{dx}{dt} + x\frac{dy}{dt}dt =& \int_0^1 -y(t)x'(t) + x(t)y'(t) \quad \text{(By definition)} \\ =& \int_0^1 -t(-1) + (1-t)(1)dt \\ =& \int_0^1 1dt \\ =& 1 \end{aligned}$$

(b) C' is the quarter circle from $(1,0)$ to $(0,1)$

(1) **Picture**



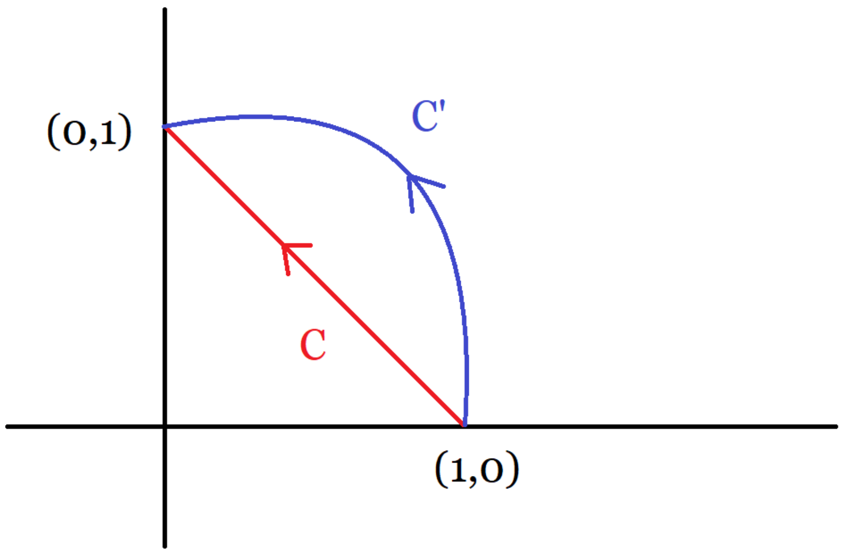
(2) **Parametrize**

$$x(t) = \cos(t)$$
$$y(t) = \sin(t)$$
$$\left(0 \leq t \leq \frac{\pi}{2}\right)$$

(3) **Integrate**

$$\begin{aligned}
\int_{C'} -ydx + xdy &= \int_0^{\frac{\pi}{2}} -y(t)x'(t) + x(t)y'(t) \\
&= \int_0^{\frac{\pi}{2}} -\sin(t)\left(-\sin(t)\right) + \cos(t)\cos(t)dt \\
&= \int_0^{\frac{\pi}{2}} 1dt \\
&= \frac{\pi}{2}
\end{aligned}$$

Remark: Note that even though C and C' have the same endpoints, we get two different answers.



In this case, we say that the line integral **depends** on the path.

Question: When is it <u>independent</u> of the path?

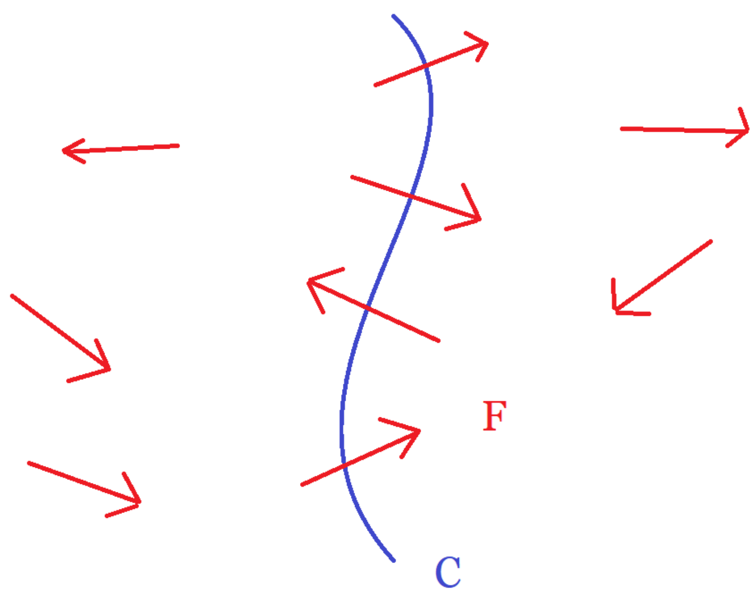
Surprisingly, this has to do with conservative vector fields! (section 16.1)

2. Line Integral of a Vector Field

Video: Line Integral of a Vector Field

Let's talk about something completely unrelated!

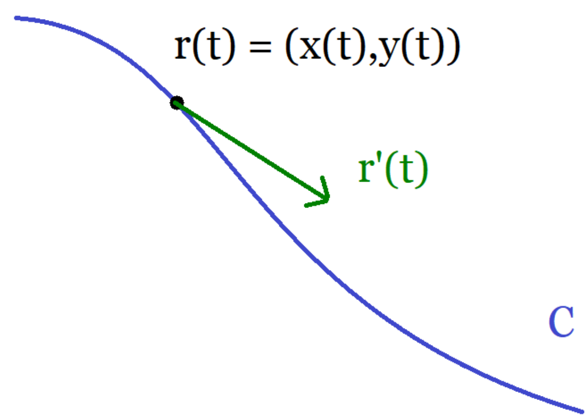
Goal: Given a vector field F and a curve C, want to sum up (= integrate) the values of F along C



(Think of collecting all the arrows as you walk along C)

Notation:

$$r(t) = (x(t), y(t))$$
$$r'(t) = \langle x'(t), y'(t) \rangle \text{ (Tangent Vector)}$$



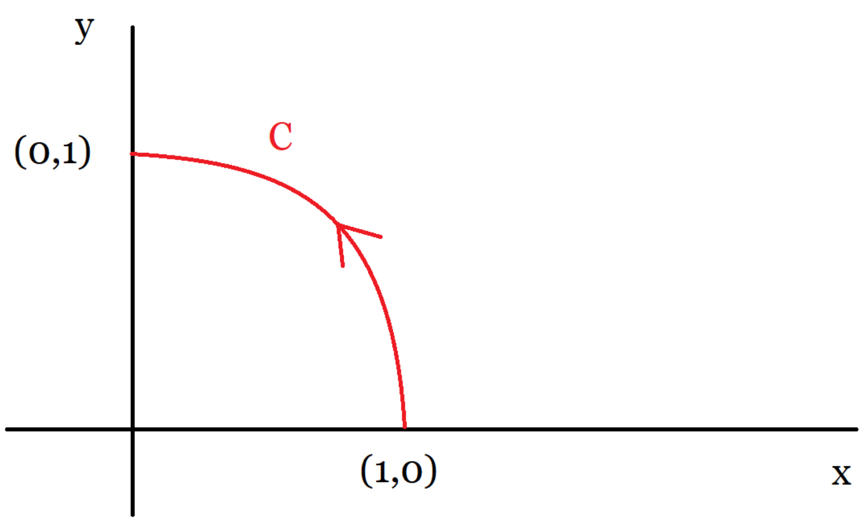
Definition: [Line Integral of F over C]

$$\int_C F \cdot dr = \int_C F \cdot \frac{dr}{dt} dt = \int_a^b F(r(t)) \cdot r'(t) dt$$

Example: $\int_C F \cdot dr$, $F(x,y) = \left\langle x^2, -xy \right\rangle$

C : Quarter Circle from $(1,0)$ to $(0,1)$

(1) **Picture**



(2) **Parametrize**

$$\begin{aligned} x(t) &= \cos(t) \\ y(t) &= \sin(t) \\ 0 \leq t \leq \frac{\pi}{2} \end{aligned}$$

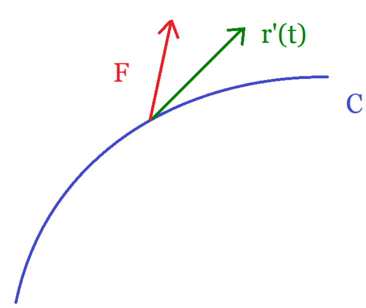
So $r(t) = (\cos(t), \sin(t))$

(3) **Integrate**

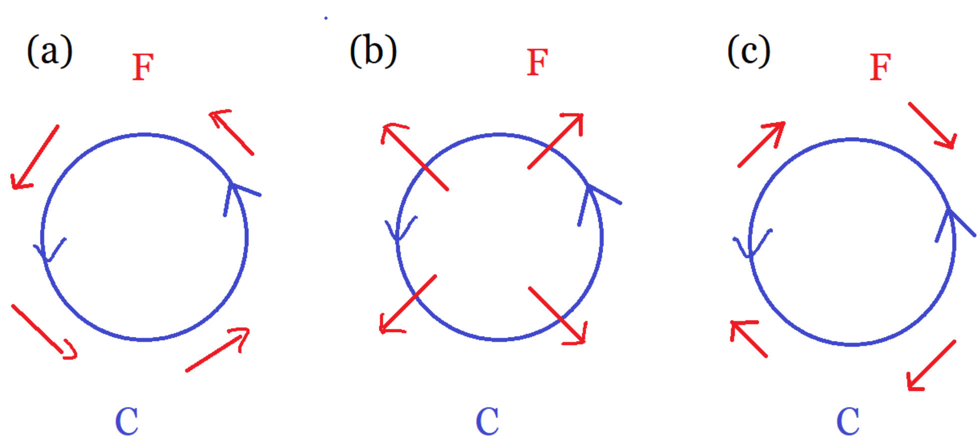
$$\begin{aligned}
\int_C F \cdot \frac{dr}{dt} dt &= \int_0^{\frac{\pi}{2}} F(r(t)) \cdot r'(t) dt \\
&= \int_0^{\frac{\pi}{2}} \left\langle (x(t))^2, -x(t)y(t) \right\rangle \cdot \langle x'(t), y'(t) \rangle \, dt \\
&= \int_0^{\frac{\pi}{2}} \left\langle \cos^2(t), -\cos(t)\sin(t) \right\rangle \cdot \langle -\sin(t), \cos(t) \rangle \, dt \\
&= \int_0^{\frac{\pi}{2}} -\cos^2(t)\sin(t) - \cos(t)\sin(t)\cos(t) dt \\
&= \int_0^{\frac{\pi}{2}} -2\cos^2(t)\sin(t) dt \\
&= \left[\frac{2}{3}\cos^3(t) \right]_0^{\frac{\pi}{2}} \\
&= -\frac{2}{3}
\end{aligned}$$

Interpretation:

(1) $F \cdot r'(t)$ is a number which measures how close F is to C, and $\int_C F \cdot dr = \int F' \cdot r'(t)$ just sums up those numbers



(2) **3 Scenarios:** In each scenario, think of running on a track C and F is wind blowing on you:



(a): $\int_C F \cdot dr$ is **BIG**
(b): $\int_C F \cdot dr = 0$ (F has no effect on C)
(c): $\int_C F \cdot dr$ is very negative

(3) If $F =$ Force, then $\int_C F \cdot dr =$ Work done of F on C

3. Connecting the two

Today we talked about two different topics: Line Integrals of a function and line integrals of vector fields. It turns out they are both the same!

Example:

$$
\begin{aligned}
\text{(Beginning of Lecture)} &= \int_C -ydx + xdy \\
&= \int_a^b -y(t)x'(t) + x(t)y'(t)dt \\
&= \int_a^b \langle -y(t), x(t) \rangle \cdot \langle x'(t), y'(t) \rangle \, dt \\
&= \int_a^b F(r(t)) \cdot r'(t) \;\; F(x,y) = \langle -y, x \rangle \\
&= \int_C F \cdot dr \\
&= \text{(End of lecture)}
\end{aligned}
$$

So both topics are just two different sides of the same coin!

Take-Away: If P and Q are functions, then

$$
\int_C Pdx + Qdy = \int_C F \cdot dr \quad \text{where } F = \langle P, Q \rangle
$$