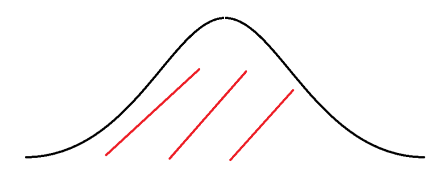
LECTURE 2: TRIPLE INTEGRALS

1. Gaussian Integral

Video: Gaussian Integral

Warning: This is the most exciting example of the course! **Nothing** in your life will be as exciting as this!

Recall: (Math 2B) e^{-x^2} does not have an antiderivative, and yet...



Example: Calculate $\int_{-\infty}^{\infty} e^{-x^2} dx$

Using Multivariable Calculus, we're going to do the impossible

(1) **Trick:** Let $I = \int_{-\infty}^{\infty} e^{-x^2} dx \geq 0$

But also $I = \int_{-\infty}^{\infty} e^{-y^2} dy$
(doesn't matter which variable we're using; potato potahto)

Date: Wednesday, January 8, 2020.

(2) **Multiply:**

$$\begin{aligned}
I^2 =& (I)(I) \\
=& \left(\int_{-\infty}^{\infty} e^{-x^2} dx\right) \left(\int_{-\infty}^{\infty} e^{-y^2} dy\right) \\
=& \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{-x^2} dx\right) e^{-y^2} dy \\
=& \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2} e^{-y^2} dx dy \\
=& \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx dy \\
=& \int_{0}^{2\pi} \int_{0}^{\infty} e^{-r^2} r dr d\theta \\
=& 2\pi \int_{0}^{\infty} r e^{-r^2} dr \\
=& 2\pi \left[\left(-\frac{1}{2}\right) e^{-r^2}\right]_{r=0}^{r=\infty} \qquad (u - sub : u = -r^2) \\
=& 2\pi \left(-\frac{1}{2} e^{-\infty} + \frac{1}{2} e^{0}\right) \\
=& 2\pi \left(\frac{1}{2}\right) \\
=& \pi
\end{aligned}$$

(3) $I^2 = \pi \Rightarrow I = \sqrt{\pi}$ (since $I > 0$)

(4) **Answer:**

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

Note: Check out this awesome playlist for 12 ways of evaluating this integral: Gaussian Integral 12 Ways

2. Triple Integrals

Now if you thought that was fun, today will be triple the fun because we'll cover triple integrals! The cool thing is that it's the exact same process: Picture, Inequality, Integrate

Example: Calculate

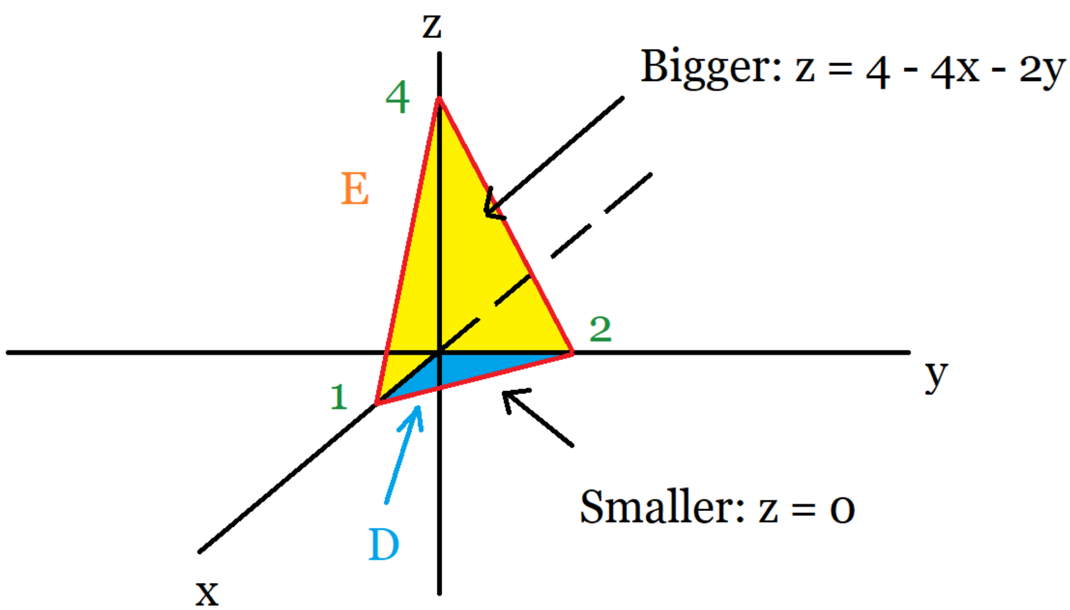
$$\int\int\int_E 6z \, dxdydz$$

where $E =$ Tetrahedron below the plane $4x+2y+z = 4$ (and $x, y \geq 0$)

(1) **Picture:** How to draw this???

Trick: To draw E, find the intercepts of $4x + 2y + z = 4$

$$\begin{array}{ll} z\text{−intercept } (x = 0, y = 0) & 4(0) + 2(0) + z = 4 \Rightarrow z = 4 \\ x\text{−intercept } (y = 0, z = 0) & 4x + 2(0) + 0 = 4 \Rightarrow x = 1 \\ y\text{−intercept } (x = 0, z = 0) & 4(0) + 2y + 0 = 4 \Rightarrow y = 2 \end{array}$$

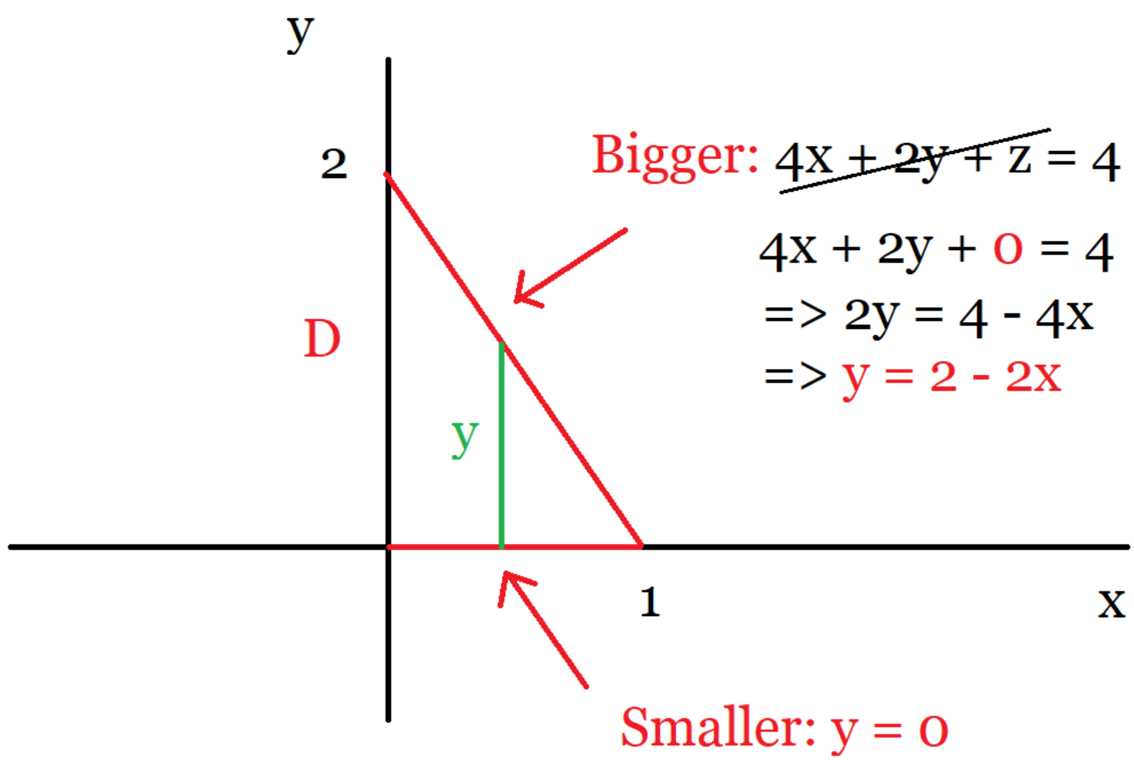


(2) This time: Smaller $\leq z \leq$ Bigger $\Rightarrow 0 \leq z \leq$ Plane
But $4x + 2y + z = 4 \Rightarrow z = 4 - 4x - 2y$
Hence $0 \leq z \leq 4 - 4x - 2y$.

(3) **Find** D

We found an inequality for z and now we need inequalities for y and x:

Notice: $z = 0$ in D



Smaller $\leq y \leq$ Bigger $\Rightarrow$ $0 \leq y \leq 2 - 2x$
Left $\leq x \leq$ Right $\Rightarrow$ $0 \leq x \leq 1$

(4) Evaluate the integral

$$
\begin{aligned}
\int\int\int_E 6z\,dx\,dy\,dz &= \int_0^1 \int_0^{2-2x} \int_0^{4-4x-2y} 6z\,dz\,dy\,dx \\
&= \int_0^1 \int_0^{2-2x} \left[3z^2\right]_{z=0}^{z=4-4x-2y} dy\,dx \\
&= \int_0^1 \int_0^{2-2x} 3(4-4x-2y)^2 dy\,dx \\
&\quad (\text{Think } (C-2y)^2) \\
&= \int_0^1 \left[3\left(\frac{1}{-2}\right)\left(\frac{1}{3}\right)(4-4x-2y)^3\right]_{y=0}^{y=2-2x} dx \\
&= -\frac{1}{2}\int_0^1 \cancel{(4-4x-2(2-2x))^3} - (4-4x-2(0))^3 dx \\
&= \frac{1}{2}\int_0^1 (4-4x)^3 dx \\
&= \cdots \\
&= 8
\end{aligned}
$$

Warning: A triple integral generally doesn't calculate a volume (unless the function is 1).

Interpretation 1: The *hyper*volume (4 dimensional volume) under $6z$ and over E is 8.

Interpretation 2: The solid E with density $6z$ has mass 8

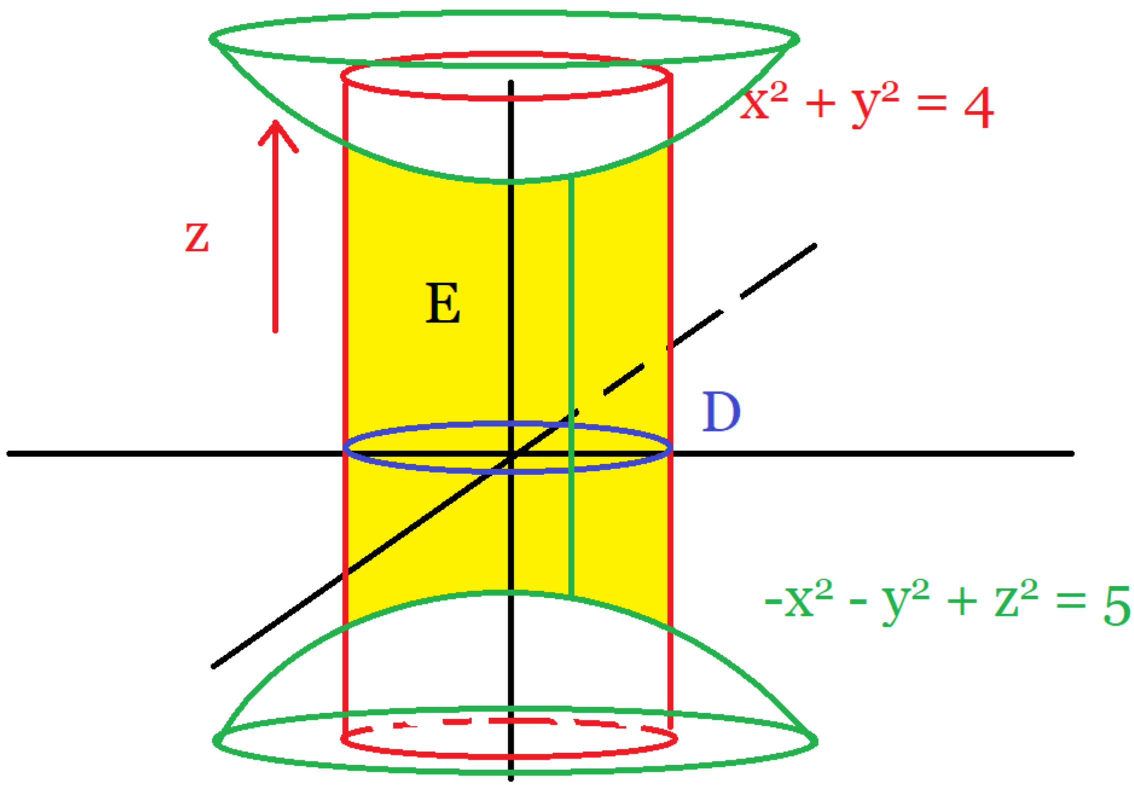
3. More Practice

Example: $\int\int\int_E 2z\,dxdydz$, where

E is the region between $x^2+y^2=4$ and $z^2-x^2-y^2=5$

Note: This question was on the midterm I gave in Fall 2018.

(1) **Picture:**
$x^2+y^2=4$ (no z here!) is a cylinder in the $z-$direction
$z^2-x^2-y^2=5$: Two minuses, hence a hyperboloid of two sheets
(= 2 cups)



(2) Small $\leq z \leq$ Big

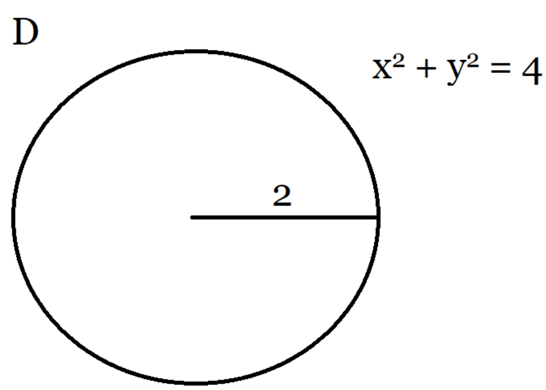
Here Small = lower part of hyperboloid and Big = upper part of hyperboloid

$$\begin{aligned} -x^2 - y^2 + z^2 &= 5 \\ \Rightarrow z^2 &= x^2 + y^2 + 5 \\ \Rightarrow z &= \pm\sqrt{x^2 + y^2 + 5} \\ \Rightarrow z &= \pm\sqrt{r^2 + 5} \end{aligned}$$

Hence: $-\sqrt{r^2+5} \leq z \leq \sqrt{r^2+5}$

(3) Draw D

Either do the usual intersection business, or just look at the picture[1] to eventually get $x^2 + y^2 = 4$, so D is a disk of radius 2.



[1] See how nice it is to have a picture?

which you can describe with $0 \leq r \leq 2, 0 \leq \theta \leq 2\pi$

(4) Integrate

$$
\begin{aligned}
\int\int\int_E 2z \, dxdydz =& \int_0^{2\pi} \int_0^2 \int_{-\sqrt{r^2+5}}^{\sqrt{r^2+5}} 2z \, r dz dr d\theta \\
=& 2\pi \int_0^2 \left[z^2 \right]_{z=-\sqrt{r^2+5}}^{z=\sqrt{r^2+5}} r dr \\
=& 2\pi \int_0^2 \left[\left(\sqrt{r^2+5} \right)^2 - \left(-\sqrt{r^2+5} \right)^2 \right] r dr \\
=& 2\pi \int_0^2 \left[\cancel{(r^2+5)} - \cancel{(r^2+5)} \right] r dr \\
=& 0
\end{aligned}
$$

Note: This is sometimes called *cylindrical coordinates*, where we do polar coordinates in x and y but don't do anything to z, more on that in section 15.7