LECTURE 2: TRIPLE INTEGRALS

1. GAUSSIAN INTEGRAL

Video: [Gaussian Integral]

Warning: This is the most exciting example of the course! Nothing in your life will be as exciting as this!

Recall: (Math 2B) $e^{-x^2}$ does not have an antiderivative, and yet...

![Gaussian Distribution](image)

Example: Calculate $\int_{-\infty}^{\infty} e^{-x^2} dx$

Using Multivariable Calculus, we’re going to do the impossible

(1) Trick: Let $I = \int_{-\infty}^{\infty} e^{-x^2} dx \geq 0$

But also $I = \int_{-\infty}^{\infty} e^{-y^2} dy$

(doesn’t matter which variable we’re using; potato potahto)

Date: Wednesday, January 8, 2020.
(2) Multiply:

\[
I^2 = (I)(I) \\
= \left( \int_{-\infty}^{\infty} e^{-x^2} \, dx \right) \left( \int_{-\infty}^{\infty} e^{-y^2} \, dy \right) \\
= \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} e^{-x^2} \, dx \right) e^{-y^2} \, dy \\
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2} e^{-y^2} \, dx \, dy \\
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} \, dx \, dy \\
= \int_{0}^{2\pi} \int_{0}^{\infty} e^{-r^2} r \, dr \, d\theta \\
= 2\pi \int_{0}^{\infty} re^{-r^2} \, dr \\
= 2\pi \left[ \left( -\frac{1}{2} \right) e^{-r^2} \right]_{r=0}^{r=\infty} \quad (u \text{ - sub: } u = -r^2)
\]

\[
= 2\pi \left( -\frac{1}{2} e^{-\infty} + \frac{1}{2} e^{0} \right) \\
= 2\pi \left( \frac{1}{2} \right) \\
= \pi
\]

(3) \( I^2 = \pi \Rightarrow I = \sqrt{\pi} \) (since \( I > 0 \))

(4) Answer:
\[ \int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi} \]

**Note:** Check out this awesome playlist for 12 ways of evaluating this integral: [Gaussian Integral 12 Ways](#)

### 2. **Triple Integrals**

Now if you thought that was fun, today will be triple the fun because we’ll cover triple integrals! The cool thing is that it’s the exact same process: Picture, Inequality, Integrate

**Example:** Calculate

\[ \iiint_E 6z \, dxdydz \]

where \( E = \text{Tetrahedron below the plane } 4x + 2y + z = 4 \text{ (and } x, y \geq 0) \)

1. **Picture:** How to draw this???

   **Trick:** To draw \( E \), find the intercepts of \( 4x + 2y + z = 4 \)

   - \( z \)-intercept \((x = 0, y = 0)\) \quad 4(0) + 2(0) + z = 4 \Rightarrow z = 4
   - \( x \)-intercept \((y = 0, z = 0)\) \quad 4x + 2(0) + 0 = 4 \Rightarrow x = 1
   - \( y \)-intercept \((x = 0, z = 0)\) \quad 4(0) + 2y + 0 = 4 \Rightarrow y = 2
(2) This time: Smaller \leq z \leq Bigger \Rightarrow 0 \leq z \leq Plane  
But 4x + 2y + z = 4 \Rightarrow z = 4 - 4x - 2y  
Hence 0 \leq z \leq 4 - 4x - 2y.

(3) Find $D$  

We found an inequality for $z$ and now we need inequalities for $y$ and $x$: 

Notice: $z = 0$ in $D$
Smaller $\leq y \leq$ Bigger $\Rightarrow$ $0 \leq y \leq 2 - 2x$

Left $\leq x \leq$ Right $\Rightarrow$ $0 \leq x \leq 1$
(4) Evaluate the integral

\[
\int \int \int_E 6z \, dx \, dy \, dz = \int_0^1 \int_0^{2-2x} \int_0^{4-4x-2y} 6z \, dz \, dy \, dx
\]

\[
= \int_0^1 \int_0^{2-2x} \left[3z^2\right]_{z=4-4x-2y}^{z=0} \, dy \, dx
\]

\[
= \int_0^1 \int_0^{2-2x} 3(4-4x-2y)^2 \, dy \, dx
\]

(Think \((C - 2y)^2\))

\[
= \int_0^1 \left[3 \left(\frac{1}{-2}\right) \left(\frac{1}{3}\right) (4-4x-2y)^3\right]_{y=0}^{y=2-2x} \, dx
\]

\[
= -\frac{1}{2} \int_0^1 (4-4x-2(2-2x))^3 - (4-4x-2(0))^3 \, dx
\]

\[
= \frac{1}{2} \int_0^1 (4-4x)^3 \, dx
\]

\[= \ldots
\]

\[= 8
\]

**Warning:** A triple integral generally doesn’t calculate a volume (unless the function is 1).

**Interpretation 1:** The hypervolume (4 dimensional volume) under \(6z\) and over \(E\) is 8.

**Interpretation 2:** The solid \(E\) with density \(6z\) has mass 8

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3. **More Practice**
Example: \( \int \int \int_E 2z \, dx \, dy \, dz \), where

\( E \) is the region between \( x^2 + y^2 = 4 \) and \( z^2 - x^2 - y^2 = 5 \)

Note: This question was on the midterm I gave in Fall 2018.

(1) Picture:
\( x^2 + y^2 = 4 \) (no \( z \) here!) is a cylinder in the \( z \)-direction
\( z^2 - x^2 - y^2 = 5 \): Two minuses, hence a hyperboloid of two sheets
\( (= 2 \text{ cups}) \)
(2) Small $\leq z \leq$ Big

Here Small = lower part of hyperboloid and Big = upper part of hyperboloid

\[-x^2 - y^2 + z^2 = 5\]
\[\Rightarrow z^2 = x^2 + y^2 + 5\]
\[\Rightarrow z = \pm \sqrt{x^2 + y^2 + 5}\]
\[\Rightarrow z = \pm \sqrt{r^2 + 5}\]

Hence: $-\sqrt{r^2 + 5} \leq z \leq \sqrt{r^2 + 5}$

(3) Draw D

Either do the usual intersection business, or just look at the picture\(^1\) to eventually get $x^2 + y^2 = 4$, so $D$ is a disk of radius 2.

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\(^1\)See how nice it is to have a picture?
which you can describe with $0 \leq r \leq 2, 0 \leq \theta \leq 2\pi$

(4) Integrate

$$\int \int \int_E 2z \, dx \, dy \, dz = \int_0^{2\pi} \int_0^2 \int_{-\sqrt{r^2+5}}^{\sqrt{r^2+5}} 2z \, r \, dz \, dr \, d\theta$$

$$= 2\pi \int_0^2 \left[ z^2 \right]_{z=-\sqrt{r^2+5}}^{z=\sqrt{r^2+5}} r \, dr$$

$$= 2\pi \int_0^2 \left[ \left( \sqrt{r^2+5} \right)^2 - \left( -\sqrt{r^2+5} \right)^2 \right] r \, dr$$

$$= 2\pi \int_0^2 \left[ (r^2+5) - (r^2+5) \right] r \, dr$$

$$= 0$$

**Note:** This is sometimes called *cylindrical coordinates*, where we do polar coordinates in $x$ and $y$ but don’t do anything to $z$, more on that in section 15.7