LECTURE 3: TRIPLE INTEGRALS (II)

Today is all about more practice with triple integrals!

Again, our mantra is: Picture, Inequalities, Math (PIM method)

1. Other directions

From the creator of *One Direction* comes a new band called *Other directions*.

Example: Calculate $\int\int\int_E 3\, dxdydx$, where

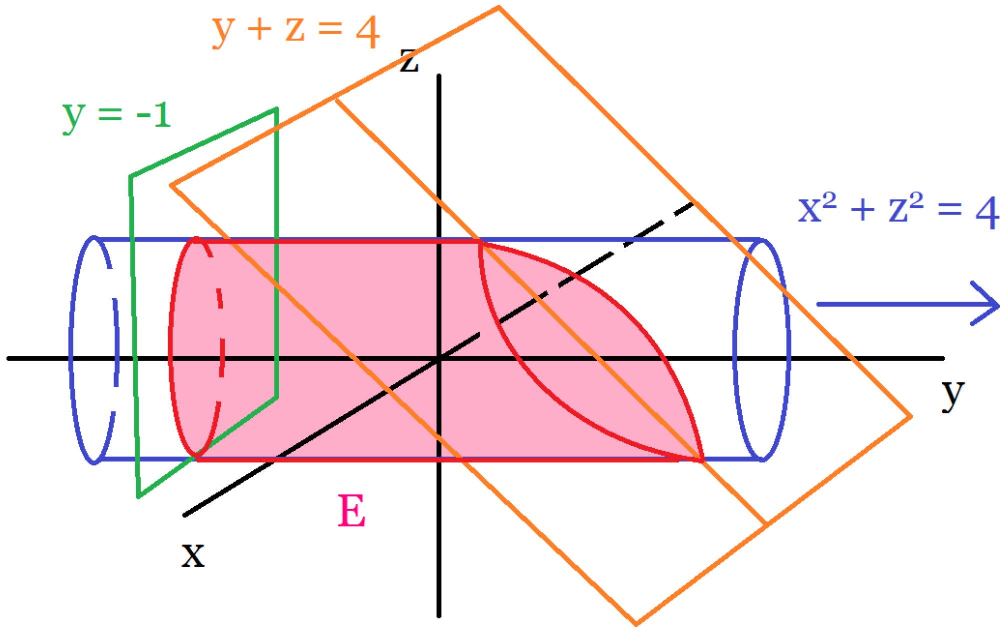
E is the solid enclosed by $x^2 + z^2 = 4, y = -1$, and $y + z = 4$

(1) **Picture:**

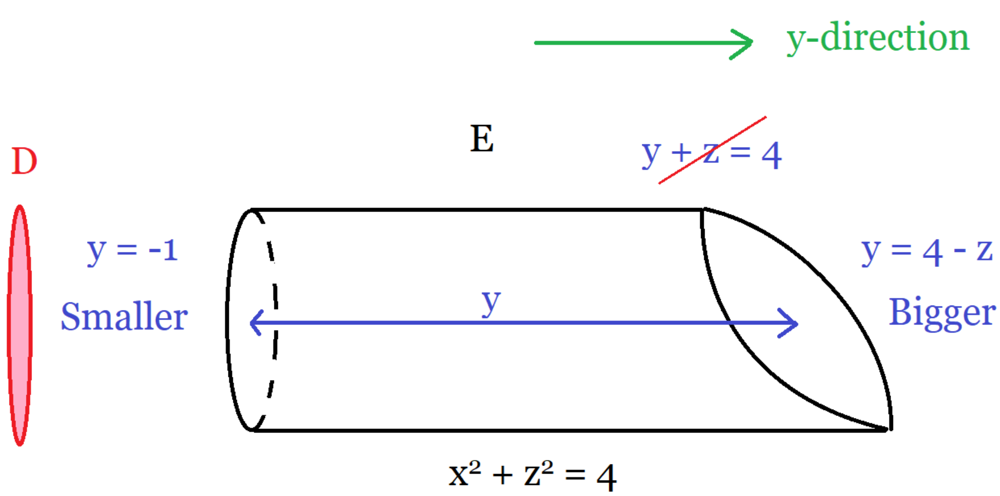
$x^2 + z^2 = 4$ is a cylinder, but in the **y**-direction (because y is missing)

$y + z = 4$ is a plane, but in the x-direction (to draw this, draw the line $y + z = 4$ and move it along the x axis)

Date: Friday, January 10, 2020.



Better picture:



(Book calls it Type 2 region)

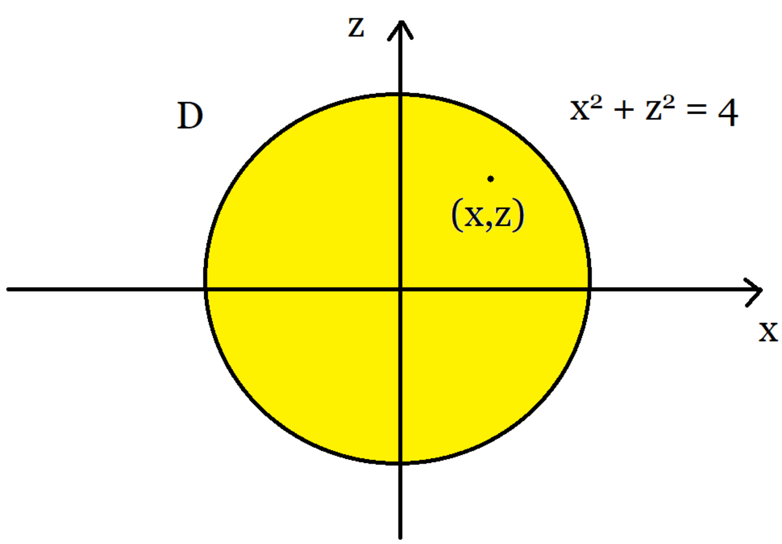
(2) **Inequalities:** Now it's *usually* Small $\leq z \leq$ Big, but since everything is in the $y-$direction, this time it's:

Small $\leq y \leq$ Big $\Rightarrow -1 \leq y \leq 4 - z$

Note: To see this, just tilt your head and see which function is above and below you!

(3) **Find** D

D is still the shadow below the surface, but this time in the y-direction. So D is a disk of radius 2 in x and z

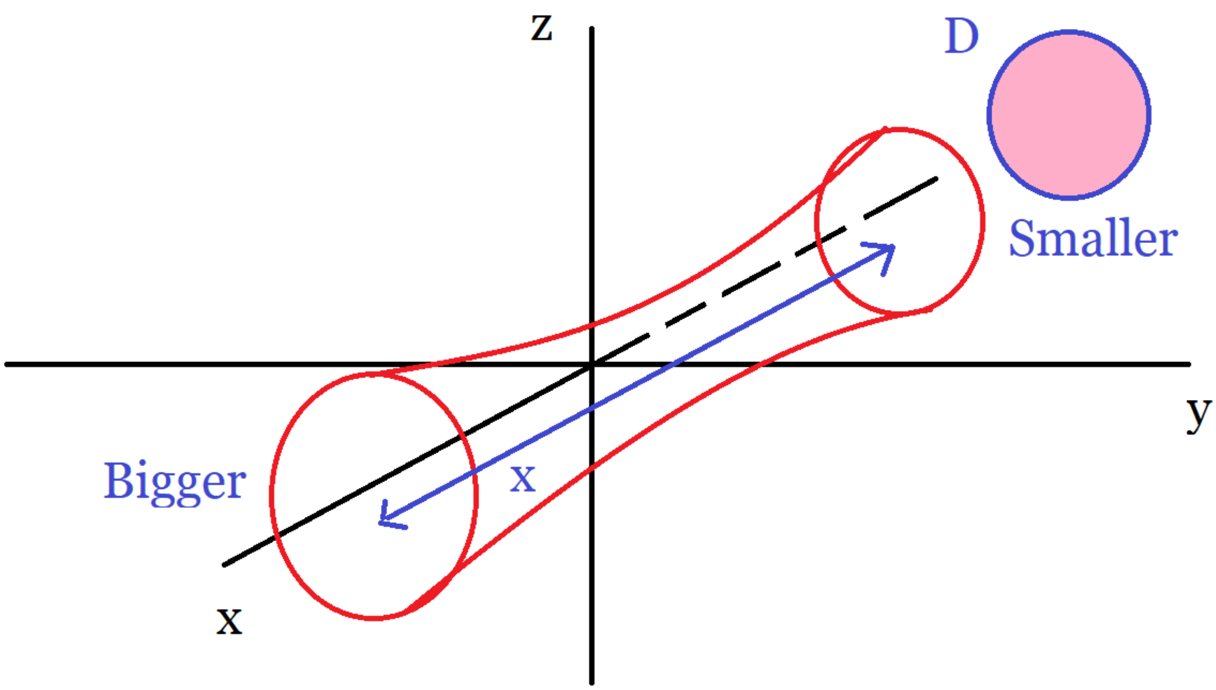


Hence we get $0 \leq r \leq 2$ and $0 \leq \theta \leq 2\pi$ In particular $x = r\cos(\theta), z = r\sin(\theta)$

(4) **Integrate** (Math)

$$
\begin{aligned}
\int\int\int_E 3\, dxdydz &= \int_0^{2\pi}\int_0^2\int_{-1}^{4-z} 3\, r dydrd\theta \\
&= \int_0^{2\pi}\int_0^2\int_{-1}^{4-r\sin(\theta)} 3r\, dydrd\theta \text{ (Use } z = r\sin(\theta)) \\
&= \int_0^{2\pi}\int_0^2 3r\left[4 - r\sin(\theta) - (-1)\right] drd\theta \\
&\quad (r \text{ doesn't depend on } y) \\
&= \int_0^{2\pi}\int_0^2 3r(5 - r\sin(\theta))drd\theta \\
&= \int_0^{2\pi}\int_0^2 15r - 3r^2\sin(\theta)drd\theta \\
&= \cdots \\
&= 60\pi
\end{aligned}
$$

Note: Sometimes your surface faces the $x-$direction, as in the following picture



In that case, the bigger function is the function in front, and the smaller one is the one in the back, and D is the shadow behind the surface.

2. Volumes

Remember that **in general** a triple integral doesn't calculate a volume, but there is one special case where it does:

Fact:

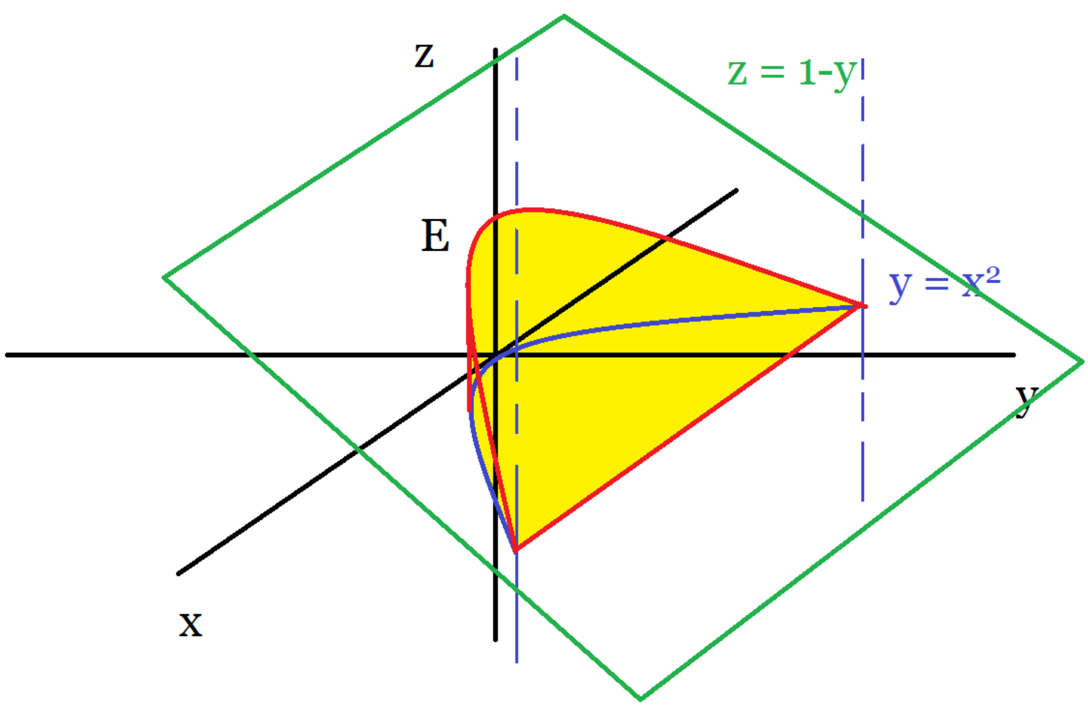
$$Vol(E) = \int\int\int_E 1 \, dxdydz$$

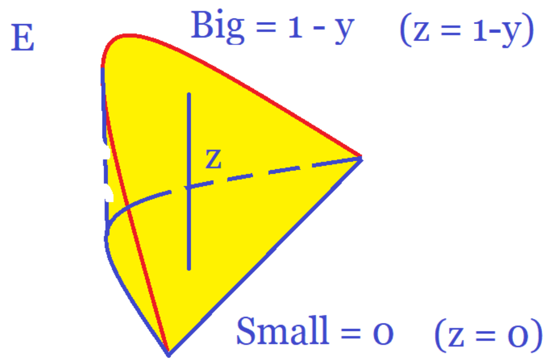
(In my opinion better to use this than double integral of bigger minus smaller)

Example Find $Vol(E)$ where E is the region enclosed by the surfaces $y = x^2, z = 0, z = 1 - y$

(1) **Picture:**

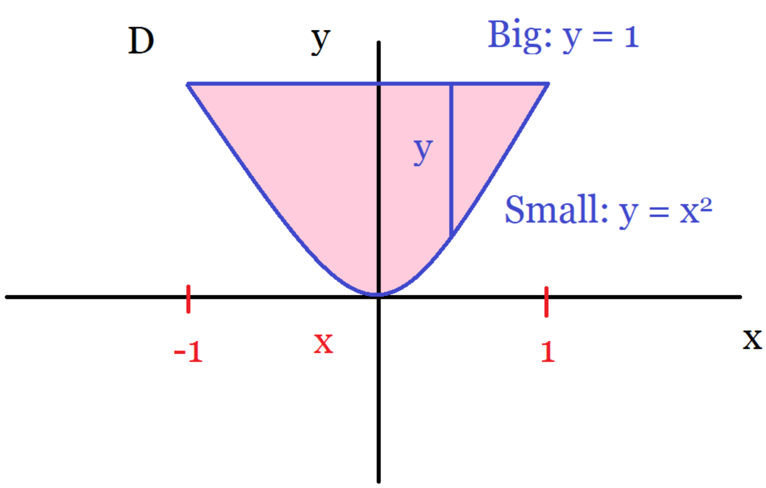
Note: $y = x^2$ (no z) is a **cylinder** in the z direction parallel to the parabola $y = x^2$. And $z = 1 - y$ (no x) is a plane in the x direction. (Visualize E as cutting a parabola along a plane)





(2) **Inequalities:** Small $\leq z \leq$ Big $\Rightarrow$ $0 \leq z \leq 1 - y$

(3) **Find** D
Note: Notice $z = 0$ in D, so $z = 1 - y \Rightarrow 0 = 1 - y \Rightarrow y = 1$



Small $\leq y \leq$ Big $\Rightarrow x^2 \leq y \leq 1$

Left $\leq x \leq$ Right $\Rightarrow -1 \leq x \leq 1$ (since $x^2 = 1 \Rightarrow x = \pm 1$)

(4) **Integrate:**

$$\begin{aligned} Vol(E) &= \int\int\int_E 1 \, dxdydz \\ &= \int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{1-y} 1 \, dzdydx \\ &= \int_{-1}^{1}\int_{x^2}^{1} 1 - y \, dydx \\ &= \cdots \\ &= \frac{8}{15} \end{aligned}$$

Warning: For volume questions shouldn't get 0 or a negative answer!

3. Intersection of two cylinders

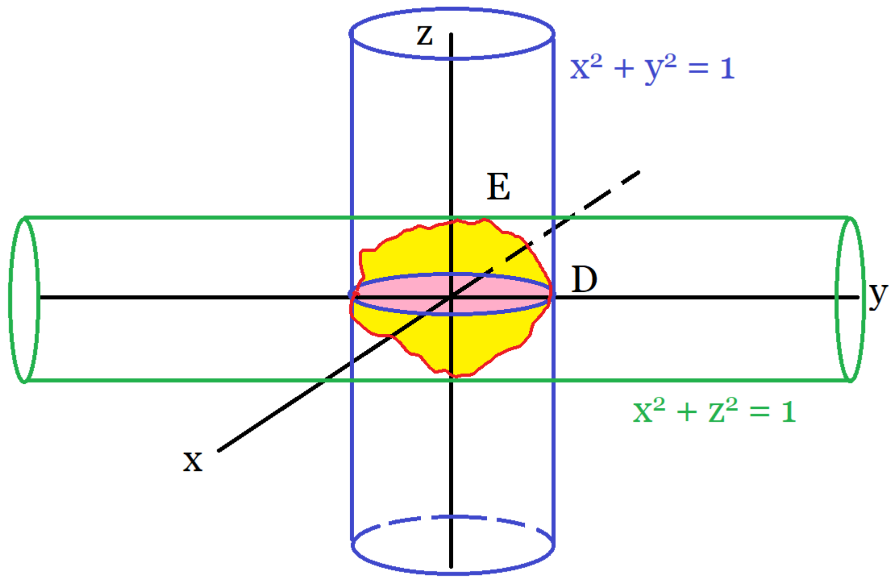
Video: Intersection of two cylinders

Last but not least, let me give you a challenge problem that math can sometimes solve things our eyes cannot see!

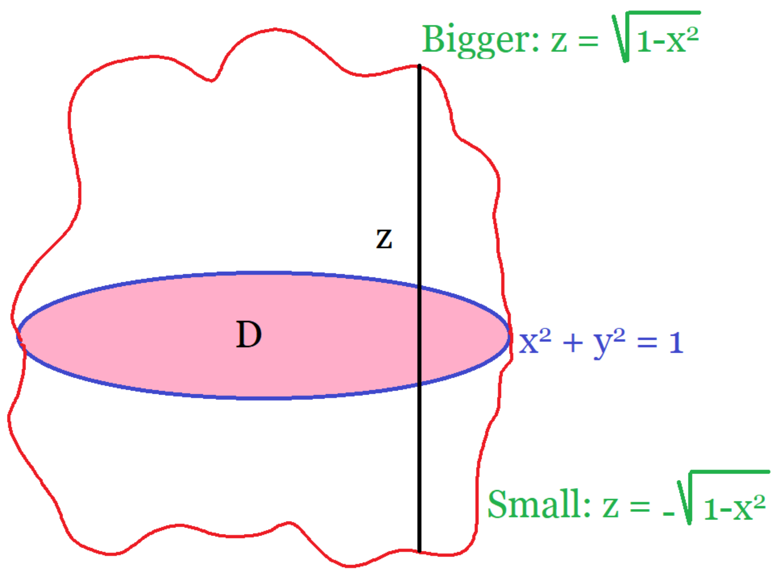
Example: Find the volume of the intersection of the cylinders $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$

(1) **Picture:**

$x^2+y^2 = 1$ (no z) is a cylinder in the $z-$direction, and $x^2+z^2 = 1$ (no x) is a cylinder in the $y-$direction.



Problem: E is hella hard to visualize! In that case: Believe in the math, not your eyes!



(2) **Inequalities:**

Smaller $\leq z \leq$ Bigger

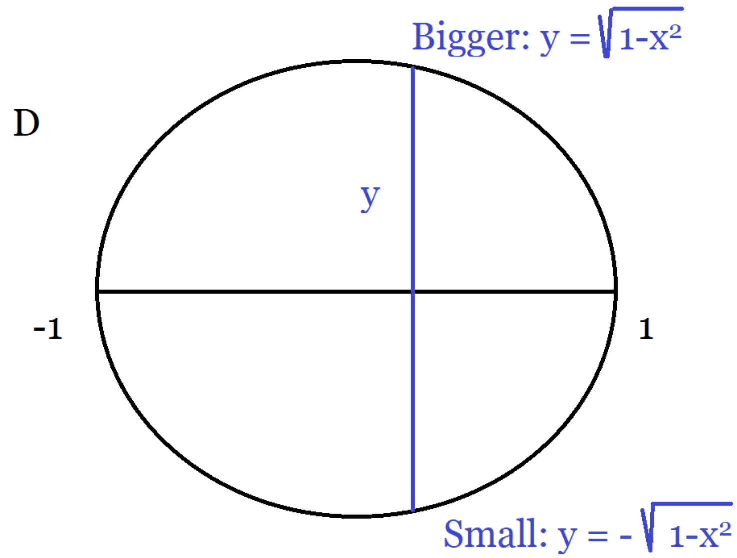
$$z^2 + x^2 = 1 \Rightarrow z^2 = 1 - x^2 \Rightarrow z = \pm\sqrt{1 - x^2}$$

Small $= -\sqrt{1-x^2}$ and Big $= \sqrt{1-x^2}$, and so
$-\sqrt{1-x^2} \leq z \leq \sqrt{1-x^2}$

Note: Why use $z^2 + x^2 = 1$? It's the only equation with z! Also it makes sense in terms of the first picture and it's the direction that makes D the easiest.

(3) **Find** D

Based on the pictures above, D is a disk of radius 1 (you can get that by setting $z = 0$ in $x^2 + y^2 = 1$)



Warning: You *could* use polar coordinates here, but if you do that (and I invite you to try it out), it becomes a **HUGE** mess, so instead go back to the bigger and smaller technique: Smaller $\leq y \leq$ Bigger

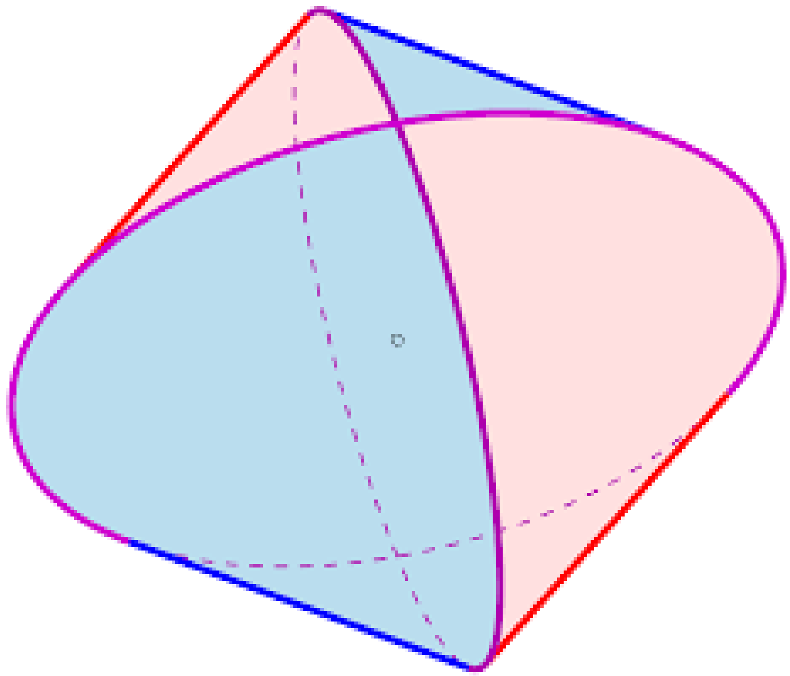
$$x^2 + y^2 = 1 \Rightarrow y^2 = 1 - x^2 \Rightarrow -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}$$

So $\boxed{-\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}}$ and $\boxed{-1 \leq x \leq 1}$

(4) **Integrate:**

$$\begin{aligned}
Vol(E) &= \int\int\int_E 1\, dxdydz \\
&= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dzdydx \\
&= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1-x^2} - \left(-\sqrt{1-x^2}\right) dydx \\
&= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 2\sqrt{1-x^2}\, dydx \\
&= \int_{-1}^{1} 2\sqrt{1-x^2} \left(\sqrt{1-x^2} - \left(-\sqrt{1-x^2}\right)\right) dx \\
&= \int_{-1}^{1} 2\sqrt{1-x^2} 2\sqrt{1-x^2} dx \\
&= \int_{-1}^{1} 4(1-x^2) dx \\
&= \frac{16}{3}
\end{aligned}$$

Note: In case you're curious what E looks like, here's a picture:



Optional: If you're even more curious: Volume of Intersection of 3 Cylinders