

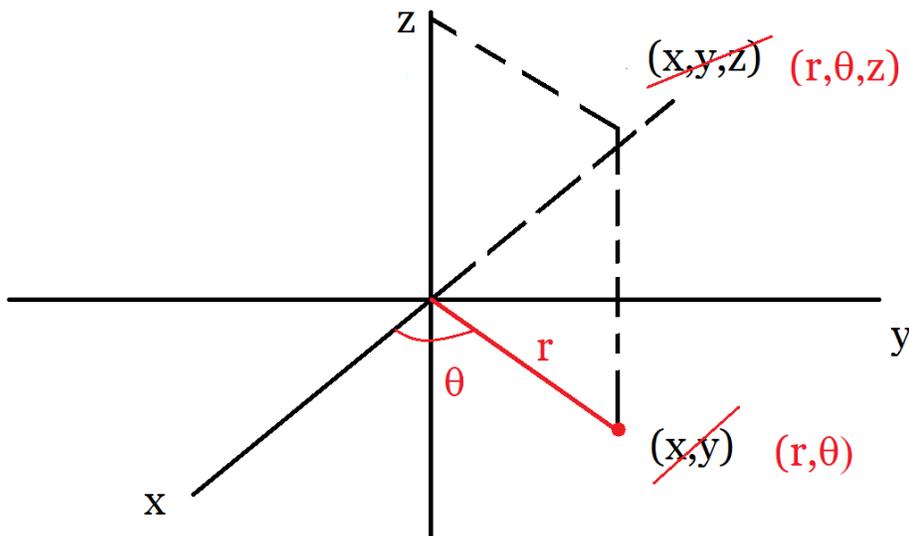
LECTURE 4: CYLINDRICAL COORDINATES

Today: We'll learn about an exciting new coordinate system called cylindrical coordinates, which helps us evaluate triple integrals.

Note: Strictly speaking we won't learn anything new: We'll still use the same polar coordinates trick, but we'll just give it a new name.

1. CYLINDRICAL COORDINATES

Basic Idea: Given (x, y, z) , apply polar coordinates to x and y , and leave z alone.



Date: Monday, January 13, 2020.

Cylindrical Coordinates:

$$x = r \cos(\theta)$$

$$y = r \sin(\theta)$$

$$z = z$$

Where:

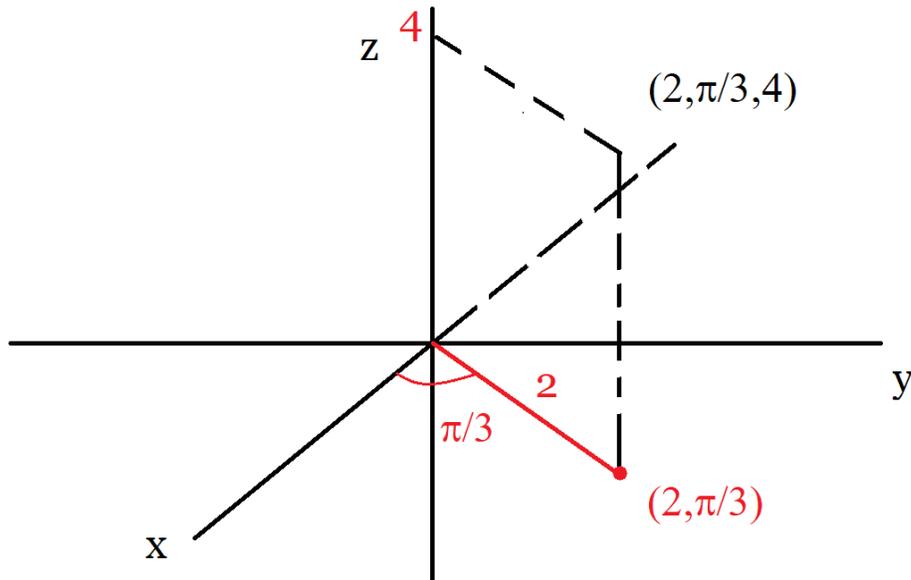
r = Distance from (x, y) to O

θ = Angle between (x, y) and x - axis

z = height of (x, y, z)

Example: Plot the point with cylindrical coordinates $(2, \frac{\pi}{3}, 4)$

$$r = 2, \theta = \frac{\pi}{3}, z = 4$$

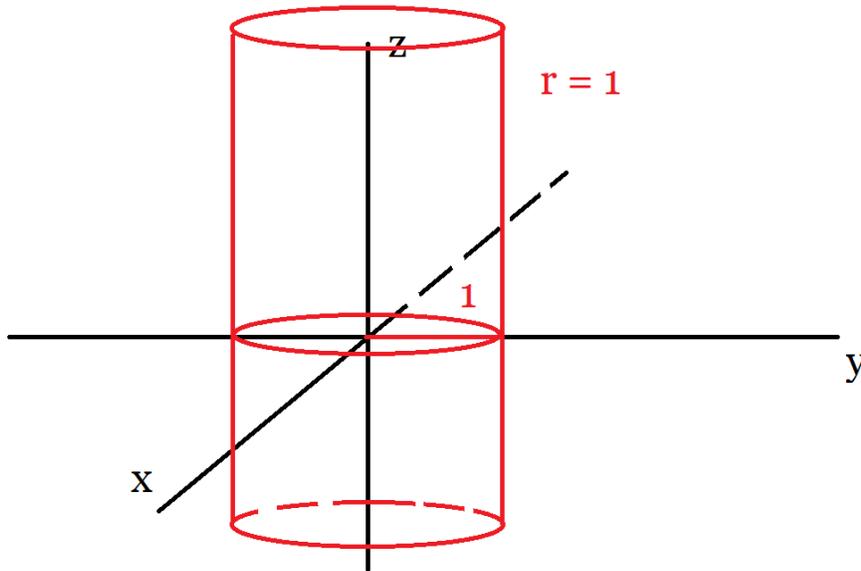


Important Fact: $x^2 + y^2 = r^2$

(This helps us simplify integrals that have $x^2 + y^2$ in them)

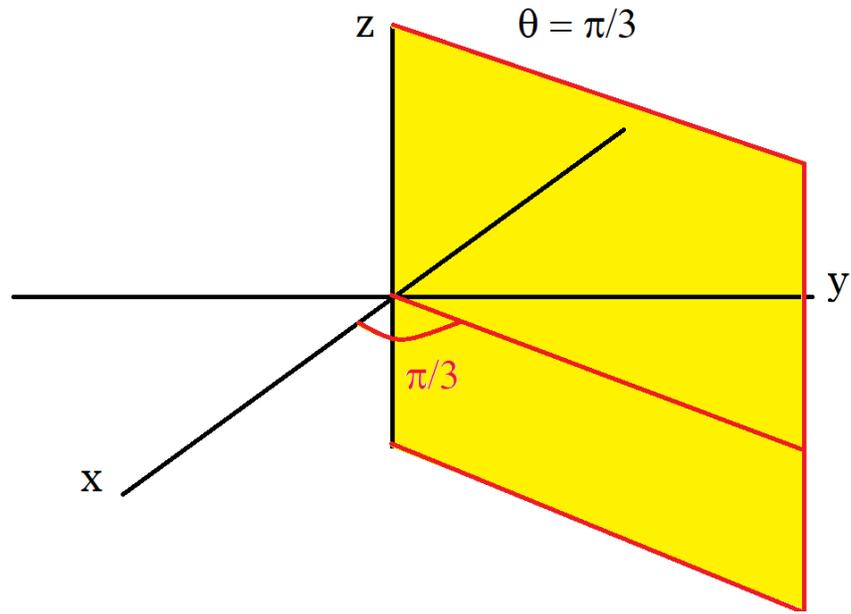
Example: Draw $r = 1$

$$r = 1 \Rightarrow \sqrt{x^2 + y^2} = 1 \Rightarrow x^2 + y^2 = 1 \text{ Cylinder}$$

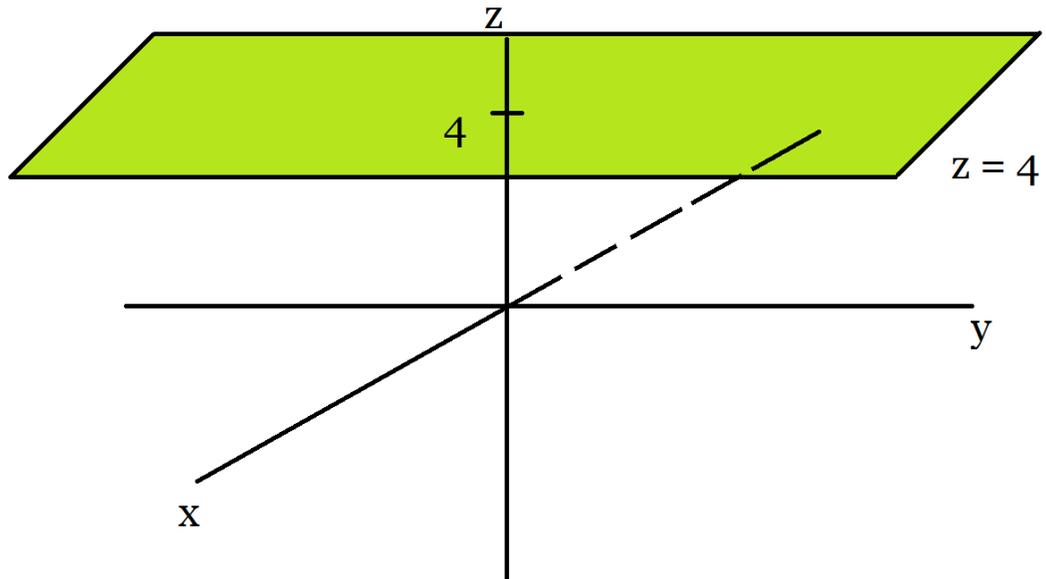


Point: $r = 1$ is much easier than $x^2 + y^2 = 1$

Example: Draw $\theta = \frac{\pi}{3}$ (and $r \geq 0$) Half-plane with angle $\frac{\pi}{3}$

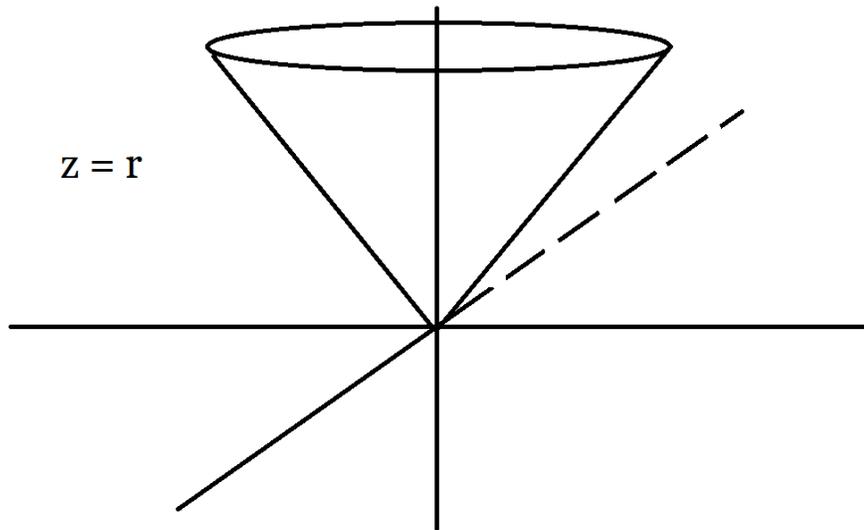


Example: Draw $z = 4$: Just the plane $z = 4$!



Example: Draw $z = r$ ($r \geq 0$)

$$z = r \Rightarrow z = \sqrt{x^2 + y^2} \Rightarrow z^2 = x^2 + y^2 \text{ Cone}$$



2. INTEGRALS IN CYLINDRICAL COORDINATES

Exactly the same thing as polar coordinates; in fact it's just the same thing we've been doing!

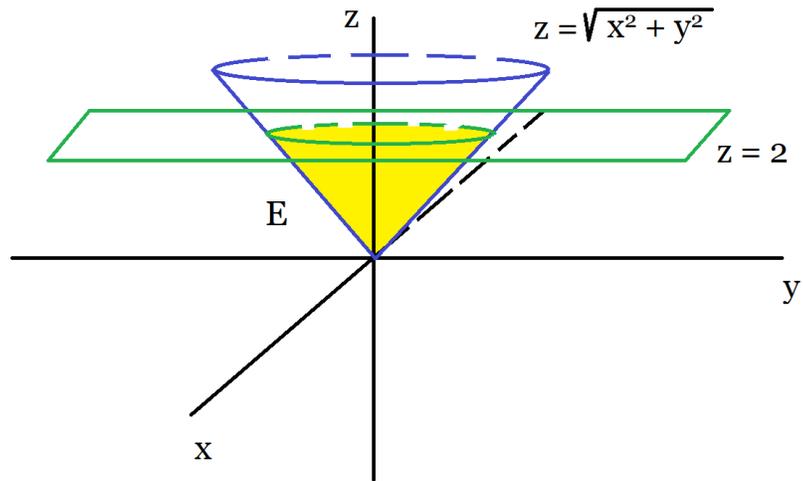
Example:

$$\int \int \int_E y^2 dx dy dz$$

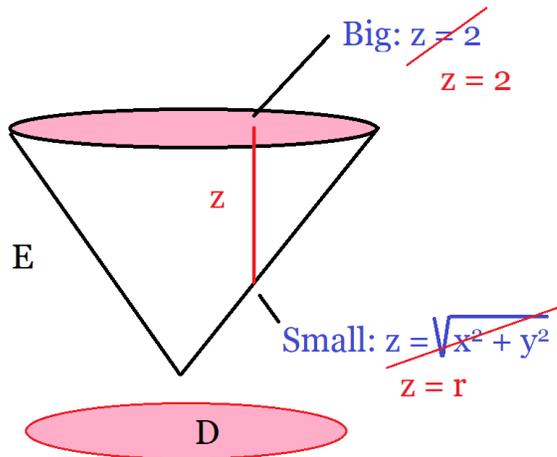
E is the region between the cone $z = \sqrt{x^2 + y^2}$ and below the plane $z = 2$.

Rule of Thumb: Use Cylindrical Coordinates whenever you see $x^2 + y^2$ or a cylinder, or a cone

(1) Picture:



Better picture:



- (2) **Inequalities:** Small $\leq z \leq$ Big $\Rightarrow r \leq z \leq 2$ (make sure to write everything in cylindrical coordinates)
- (3) **Find D :** Intersect $z = 2$ and $z = r \Rightarrow r = 2$ $D =$ disk of radius 2, so $0 \leq r \leq 2$ and $0 \leq \theta \leq 2\pi$
- (4) **Integrate** (Math)

$$\begin{aligned}
 \int \int \int_E y^2 \, dx \, dy \, dz &= \int_0^{2\pi} \int_0^2 \int_r^2 (r \sin(\theta))^2 r \, dz \, dr \, d\theta \\
 &= \int_0^{2\pi} \int_0^2 \int_r^2 r^3 \sin^2(\theta) \, dz \, dr \, d\theta \\
 &= \int_0^{2\pi} \int_0^2 (2-r)r^3 \sin^2(\theta) \, dr \, d\theta \\
 &= \left(\int_0^2 (2-r)r^3 \, dr \right) \left(\int_0^{2\pi} \sin^2(\theta) \, d\theta \right) \\
 &= \dots \\
 &= \frac{8\pi}{5}
 \end{aligned}$$

Important Reminder:

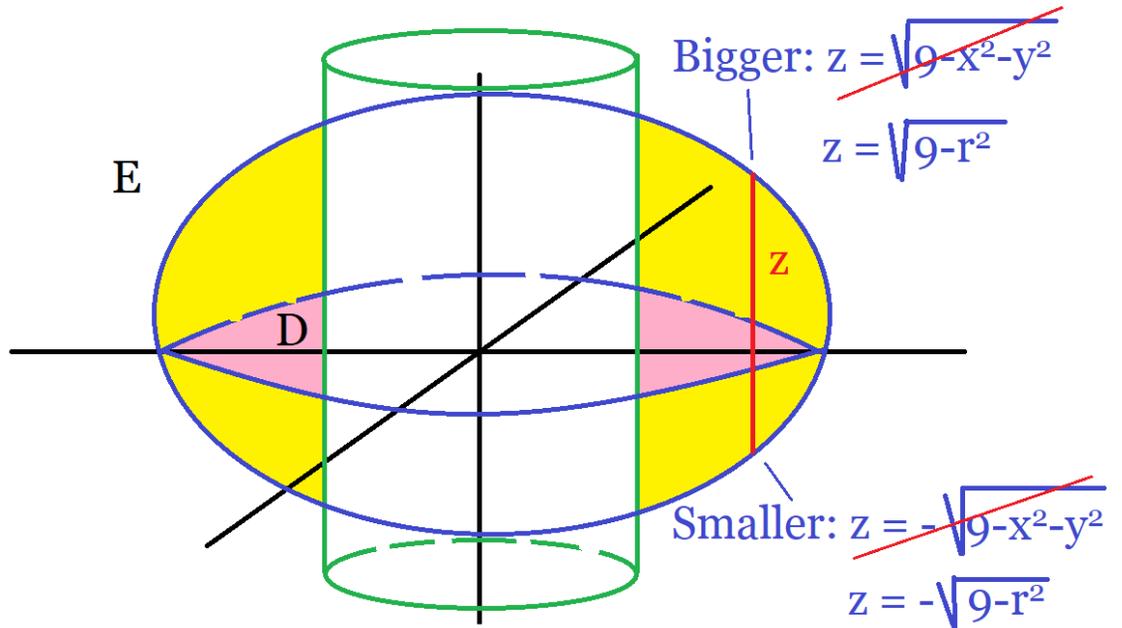
$$\begin{aligned}
 \int_0^{2\pi} \sin^2(\theta) \, d\theta &= \int_0^{2\pi} \frac{1}{2} - \frac{1}{2} \cos(2\theta) \, d\theta \\
 &= \left[\frac{\theta}{2} - \left(\frac{1}{2} \right) \left(\frac{1}{2} \right) \sin(2\theta) \right]_{\theta=0}^{\theta=2\pi} \\
 &= \frac{2\pi}{2} - \frac{1}{4} \sin(4\pi) - 0 + \frac{1}{4} \sin(0) \\
 &= \pi
 \end{aligned}$$

Example

$$\int \int \int_E \tan^{-1} \left(\frac{y}{x} \right) dx dy dz$$

where E is the solid outside $x^2 + y^2 = 4$ (cylinder) but inside $x^2 + y^2 + z^2 = 9$ (sphere)

(1) **Picture:**

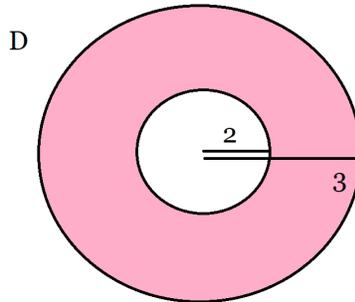


(2) **Inequalities:** Small $\leq z \leq$ Big

$$x^2 + y^2 + z^2 = 9 \Rightarrow z^2 = 9 - x^2 - y^2 \Rightarrow z^2 = 9 - r^2 \Rightarrow z = \pm \sqrt{9 - r^2}$$

$$\text{Hence: } -\sqrt{9 - r^2} \leq z \leq \sqrt{9 - r^2}$$

- (3) **Find D :** If you set $z = 0$ in $x^2 + y^2 + z^2 = 9$ you get $x^2 + y^2 = 9$, which is a circle of radius 3. But since we're focusing on the part outside the cylinder $x^2 + y^2 = 4$, D is actually a ring (annulus):



So $2 \leq r \leq 3$ and $0 \leq \theta \leq 2\pi$

- (4) **Integrate** (Math)

Note: $\tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{r \sin(\theta)}{r \cos(\theta)}\right) = \tan^{-1}(\tan(\theta)) = \theta$

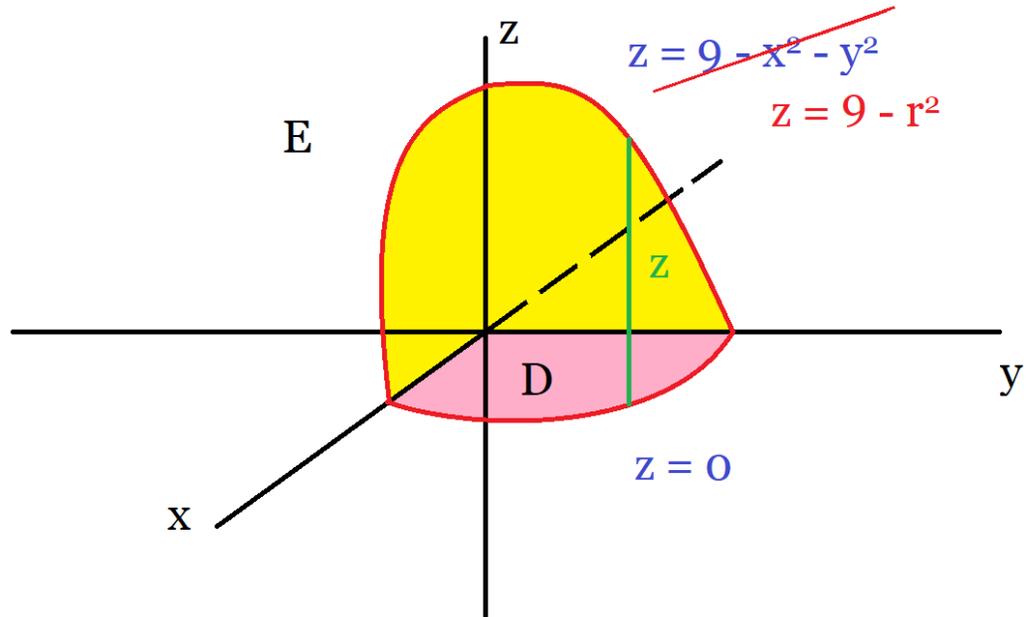
$$\begin{aligned}
\int \int \int_E \tan^{-1} \left(\frac{y}{x} \right) dx dy dz &= \int_0^{2\pi} \int_2^3 \int_{-\sqrt{9-r^2}}^{\sqrt{9-r^2}} \theta r dz dr d\theta \\
&= \int_0^{2\pi} \int_2^3 \theta r \left(\sqrt{9-r^2} - \left(-\sqrt{9-r^2} \right) \right) dr d\theta \\
&= \int_0^{2\pi} \int_2^3 \theta 2r \sqrt{9-r^2} dr d\theta \\
&= \left(\int_0^{2\pi} \theta d\theta \right) \left(\int_2^3 2r(9-r^2)^{\frac{1}{2}} dr \right) \\
&= \left[\frac{\theta^2}{2} \right]_0^{2\pi} \left[-\frac{2}{3}(9-r^2)^{\frac{3}{2}} \right]_{r=2}^{r=3} \\
&= \dots \\
&= \frac{20}{3} \pi^2 \sqrt{5}
\end{aligned}$$

Example: (if time permits)

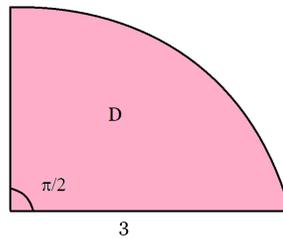
$$\int \int \int_E z dx dy dz$$

E is the solid in the first octant under $z = 9 - x^2 - y^2$

(1) **Picture:**



- (2) **Inequalities:** Small $\leq z \leq$ Big $\Rightarrow 0 \leq z \leq 9 - r^2$
- (3) **Find D :** Set $z = 0$ in $z = 9 - r^2 \Rightarrow r^2 = 9 \Rightarrow r = 3$
Warning: But since we're in the first octant, D is actually a quarter disk!



So $0 \leq r \leq 3$ and $0 \leq \theta \leq \frac{\pi}{2}$

(4) **Integrate** (Math)

$$\begin{aligned}\iint\int_E z \, dx \, dy \, dz &= \int_0^{\frac{\pi}{2}} \int_0^3 \int_0^{9-r^2} z \, r \, dz \, dr \, d\theta \\ &= \frac{\pi}{2} \int_0^3 \left[\frac{z^2}{2} \right]_{z=0}^{z=9-r^2} r \, dr \\ &= \frac{\pi}{2} \int_0^3 \frac{1}{2} (9-r^2)^2 r \, dr \\ &= \frac{\pi}{4} \int_0^3 81r - 18r^3 + r^5 \, dr \\ &= \dots \\ &= \frac{243\pi}{8}\end{aligned}$$