LECTURE 4: CYLINDRICAL COORDINATES

Today: We’ll learn about an exciting new coordinate system called cylindrical coordinates, which helps us evaluate triple integrals.

Note: Strictly speaking we won’t learn anything new: We’ll still use the same polar coordinates trick, but we’ll just give it a new name.

1. CYLINDRICAL COORDINATES

Basic Idea: Given \((x, y, z)\), apply polar coordinates to \(x\) and \(y\), and leave \(z\) alone.

Date: Monday, January 13, 2020.
Cylindrical Coordinates:

\[
\begin{align*}
    x &= r \cos(\theta) \\
    y &= r \sin(\theta) \\
    z &= z
\end{align*}
\]

Where:

- \( r \) = Distance from \((x, y)\) to \( O \)
- \( \theta \) = Angle between \((x, y)\) and \( x - \text{axis} \)
- \( z \) = height of \((x, y, z)\)

**Example:** Plot the point with cylindrical coordinates \((2, \frac{\pi}{3}, 4)\)

\( r = 2, \theta = \frac{\pi}{3}, z = 4 \)
Important Fact: $x^2 + y^2 = r^2$
(This helps us simplify integrals that have $x^2 + y^2$ in them)

Example: Draw $r = 1$

$$r = 1 \Rightarrow \sqrt{x^2 + y^2} = 1 \Rightarrow x^2 + y^2 = 1 \text{ Cylinder}$$

Point: $r = 1$ is much easier than $x^2 + y^2 = 1$

Example: Draw $\theta = \frac{\pi}{3}$ (and $r \geq 0$) Half-plane with angle $\frac{\pi}{3}$
Example: Draw $z = 4$: Just the plane $z = 4$!
Example: Draw $z = r$ ($r \geq 0$)

\[ z = r \Rightarrow z = \sqrt{x^2 + y^2} \Rightarrow z^2 = x^2 + y^2 \text{ Cone} \]

2. Integrals in Cylindrical Coordinates

Exactly the same thing as polar coordinates; in fact it’s just the same thing we’ve been doing!

Example:

\[ \int \int \int_E y^2 \, dxdydz \]

$E$ is the region between the cone $z = \sqrt{x^2 + y^2}$ and below the plane $z = 2$. 
Rule of Thumb: Use Cylindrical Coordinates whenever you see $x^2 + y^2$ or a cylinder, or a cone

(1) Picture:
(2) **Inequalities:** Small $\leq z \leq$ Big $\Rightarrow r \leq z \leq 2$ (make sure to write everything in cylindrical coordinates)

(3) **Find** $D$: Intersect $z = 2$ and $z = r$ $\Rightarrow r = 2$ $D =$ disk of radius 2, so $0 \leq r \leq 2$ and $0 \leq \theta \leq 2\pi$

(4) **Integrate** (Math)

\[
\int \int \int_E y^2 \, dx \, dy \, dz = \int_0^{2\pi} \int_0^2 \int_r^2 (r \sin(\theta))^2 \, r \, dz \, dr \, d\theta \\
= \int_0^{2\pi} \int_0^2 \int_r^2 r^3 \sin^2(\theta) \, dz \, dr \, d\theta \\
= \int_0^{2\pi} \int_0^2 (2 - r) r^3 \sin^2(\theta) \, dr \, d\theta \\
= \left( \int_0^2 (2 - r) r^3 \, dr \right) \left( \int_0^{2\pi} \sin^2(\theta) \, d\theta \right) \\
= \ldots \\
= \frac{8\pi}{5}
\]

**Important Reminder:**

\[
\int_0^{2\pi} \sin^2(\theta) \, d\theta = \int_0^{2\pi} \frac{1}{2} - \frac{1}{2} \cos(2\theta) \, d\theta \\
= \left[ \frac{\theta}{2} - \left( \frac{1}{2} \right) \left( \frac{1}{2} \right) \sin(2\theta) \right]_{\theta=0}^{\theta=2\pi} \\
= \frac{2\pi}{2} - \frac{1}{4} \sin(4\pi) - 0 + \frac{1}{4} \sin(0) \\
= \pi
\]

**Example**

\[ \int \int \int_{E} \tan^{-1} \left( \frac{y}{x} \right) \, dx \, dy \, dz \]

where \( E \) is the solid outside \( x^2 + y^2 = 4 \) (cylinder) but inside \( x^2 + y^2 + z^2 = 9 \) (sphere)

(1) **Picture:**

(2) **Inequalities:**  Small \( \leq z \leq \text{Big} \)

\[ x^2 + y^2 + z^2 = 9 \Rightarrow z^2 = 9 - x^2 - y^2 \Rightarrow z^2 = 9 - r^2 \Rightarrow z = \pm \sqrt{9 - r^2} \]

Hence: \( -\sqrt{9 - r^2} \leq z \leq \sqrt{9 - r^2} \)
(3) **Find** $D$: If you set $z = 0$ in $x^2 + y^2 + z^2 = 9$ you get $x^2 + y^2 = 9$, which is a circle of radius 3. But since we're focusing on the part outside the cylinder $x^2 + y^2 = 4$, $D$ is actually a ring (annulus):

So $2 \leq r \leq 3$ and $0 \leq \theta \leq 2\pi$

(4) **Integrate** (Math)

**Note:** $\tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{r \sin(\theta)}{r \cos(\theta)}\right) = \tan^{-1}(\tan(\theta)) = \theta$
\[ \int \int \int_E \tan^{-1} \left( \frac{y}{x} \right) \, dxdydz = \int_0^{2\pi} \int_0^3 \int_{\sqrt{9-r^2}}^{\sqrt{9-r^2}} \theta \, r \, dz \, dr \, d\theta \]

\[ = \int_0^{2\pi} \int_0^3 \theta r \left( \sqrt{9-r^2} - \left( -\sqrt{9-r^2} \right) \right) \, dr \, d\theta \]

\[ = \int_0^{2\pi} \int_0^3 2r \sqrt{9-r^2} \, dr \, d\theta \]

\[ = \left( \int_0^{2\pi} \theta d\theta \right) \left( \int_0^3 2r(9-r^2)^{\frac{1}{2}} \, dr \right) \]

\[ = \left[ \frac{\theta^2}{2} \right]_0^{2\pi} \left[ \frac{2}{3} (9-r^2)^{\frac{3}{2}} \right]_{r=3}^{r=0} \]

\[ = \cdots \]

\[ = \frac{20}{3} \pi^2 \sqrt{5} \]

**Example:** (if time permits)

\[ \int \int \int_E zdxdydz \]

\( E \) is the solid in the first octant under \( z = 9 - x^2 - y^2 \)

(1) **Picture:**

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> **LECTURE 4: CYLINDRICAL COORDINATES**
(2) **Inequalities:** Small ≤ z ≤ Big ⇒ 0 ≤ z ≤ 4 − r²

(3) **Find** D: Set z = 0 in z = 9 − r² ⇒ r² = 4 ⇒ r = 2
**Warning:** But since we’re in the first octant, D is actually a **quarter** disk!

So 0 ≤ r ≤ 3 and 0 ≤ θ ≤ \( \frac{\pi}{2} \)
(4) Integrate (Math)

\[ \int \int \int_E z \, dx \, dy \, dz = \int_0^\pi \int_0^3 \int_0^{9-r^2} z \, r \, dz \, dr \, d\theta \]
\[ = \frac{\pi}{2} \int_0^3 \left[ \frac{z^2}{2} \right]_{z=0}^{z=9-r^2} r \, dr \]
\[ = \frac{\pi}{2} \int_0^3 \frac{1}{2} (9 - r^2)^2 \, r \, dr \]
\[ = \frac{\pi}{4} \int_0^3 81r - 18r^3 + r^5 \, dr \]
\[ = \ldots \]
\[ = \frac{243\pi}{8} \]