LECTURE 5: SPHERICAL COORDINATES

Today's lecture is about spherical coordinates, which is *the* correct generalization of polar coordinates to three dimensions.

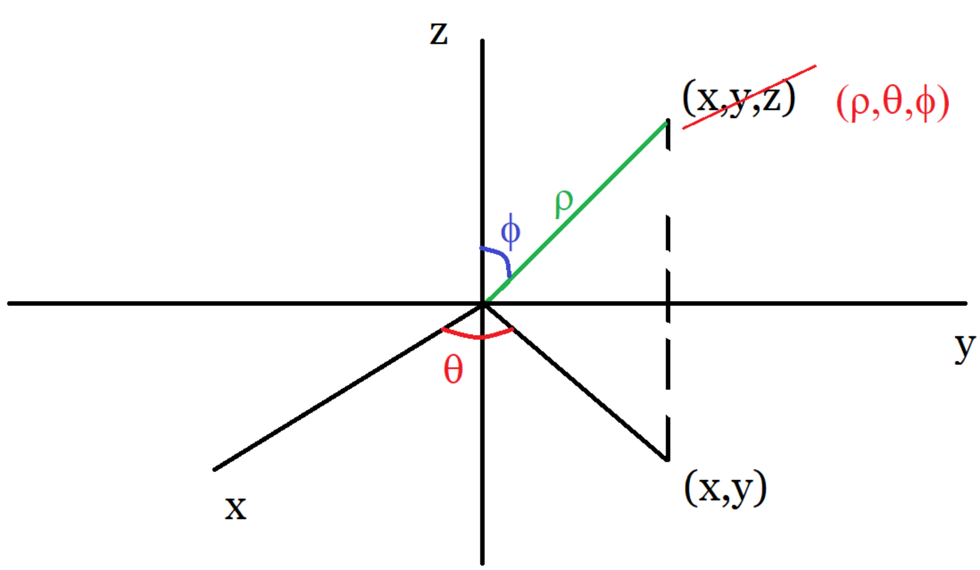
1. Spherical Coordinates

Idea: Represent points as (ρ, θ, ϕ), where:

(1) $\rho =$ distance from 0 to (x, y, z) (**RHO**dius)

(2) $\theta =$ angle between (x, y) and x-axis (**THO**rizontal)

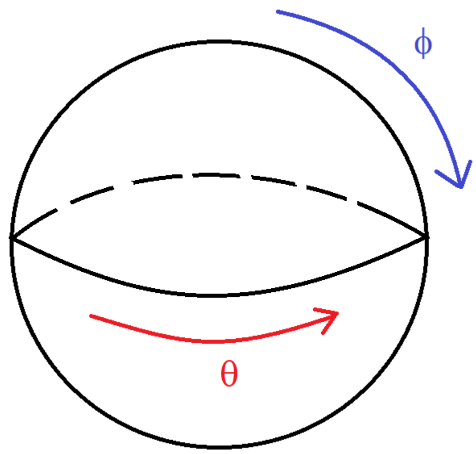
(3) $\phi =$ angle between (x, y, z) and $z-$axis (**PHE**rtical)



Date: Tuesday, January 14, 2020.

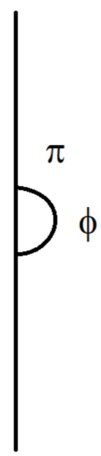
Remarks:

(1) Think of θ as a longitude and ϕ as a latitude



(2) Constraints

$$
\begin{aligned}
\rho &\geq 0 \\
0 \leq \theta &\leq 2\pi \\
0 \leq \phi &\leq \pi
\end{aligned}
$$

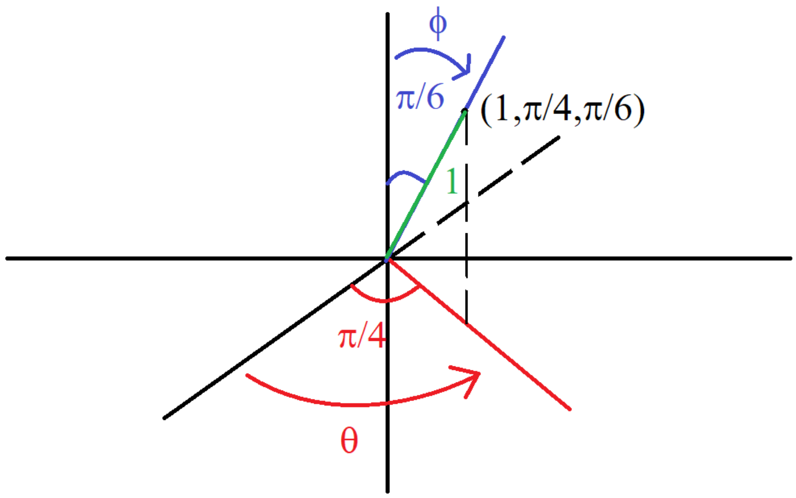


(3) Most important property: $x^2 + y^2 + z^2 = \rho^2$ (much easier!)

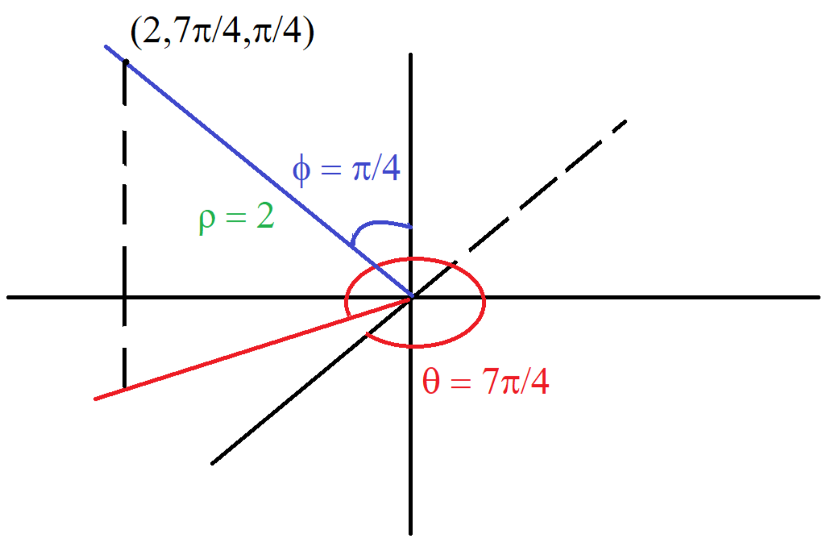
Example: Plot the following points:

(a) $\left(1, \frac{\pi}{4}, \frac{\pi}{6}\right)$

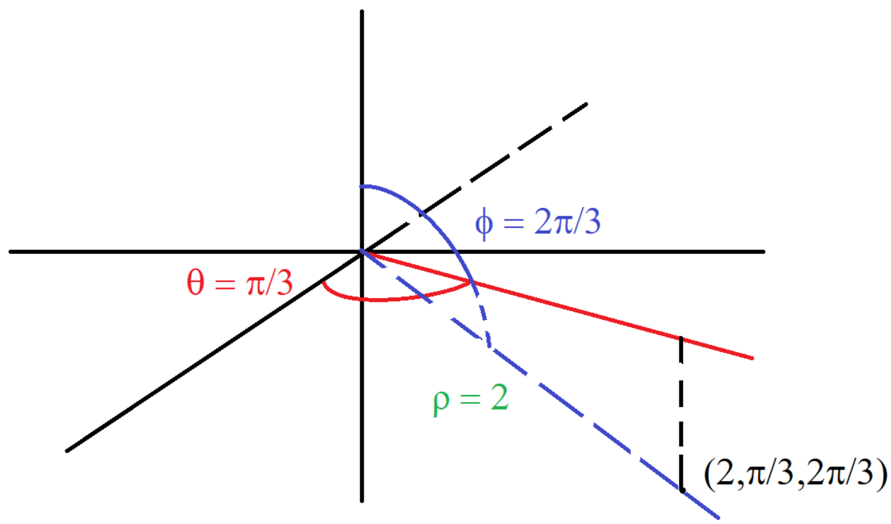
Think like the hands of a robot You move horizontally (= right) by $\frac{\pi}{4}$, starting from the $x-$axis and move vertically (= down) by $\frac{\pi}{6}$ starting from the $z-$axis.



(b) $\left(2, \frac{7\pi}{4}, \frac{\pi}{4}\right)$



(c) $\left(2, \frac{\pi}{3}, \frac{2\pi}{3}\right)$

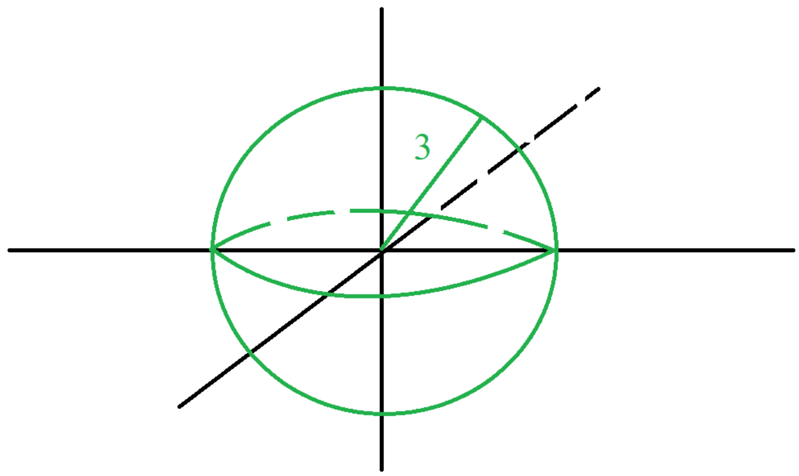


Just like last time, this is useful because a lot of familiar objects can be written really neatly in terms of spherical coordinates.

Example: Sketch the following surfaces

(a) $\rho = 3$

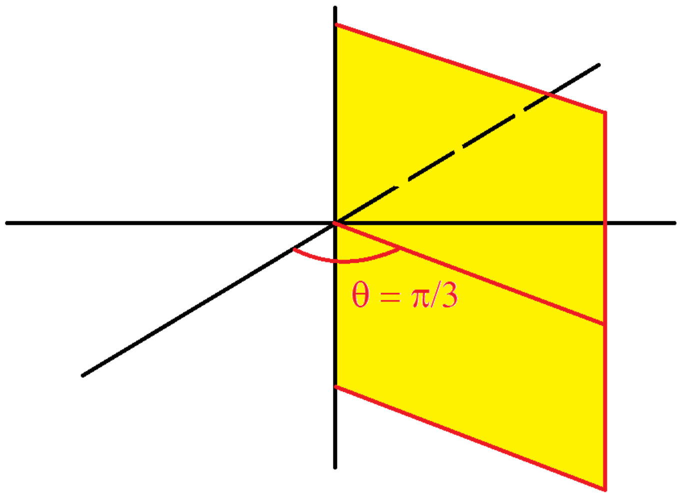
$$\sqrt{x^2 + y^2 + z^2} = 3 \Rightarrow x^2 + y^2 + z^2 = 9 \text{ Sphere}$$



Point: $\rho = 3$ is much easier than $x^2 + y^2 + z^2 = 9$

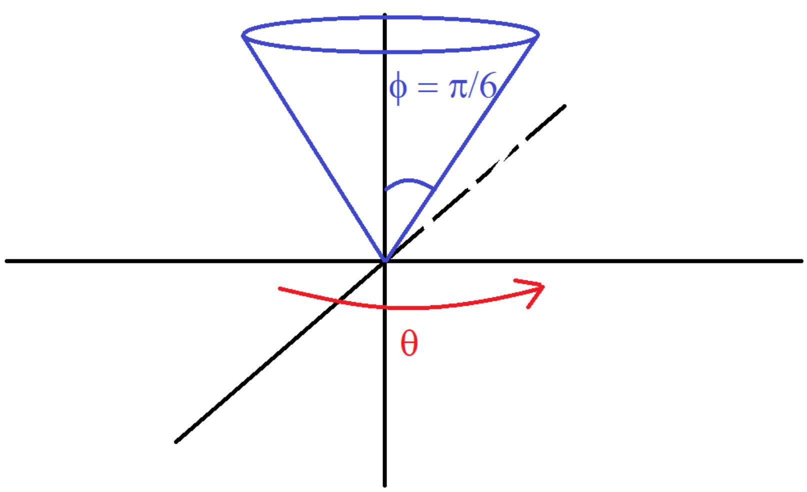
(b) $\theta = \frac{\pi}{3}$

Half-plane through $\theta = \frac{\pi}{3}$ (just like last time)



(c) $\phi = \frac{\pi}{6}$

Upper-cone!



Note: Lower cone is $\phi = \frac{5\pi}{6}$

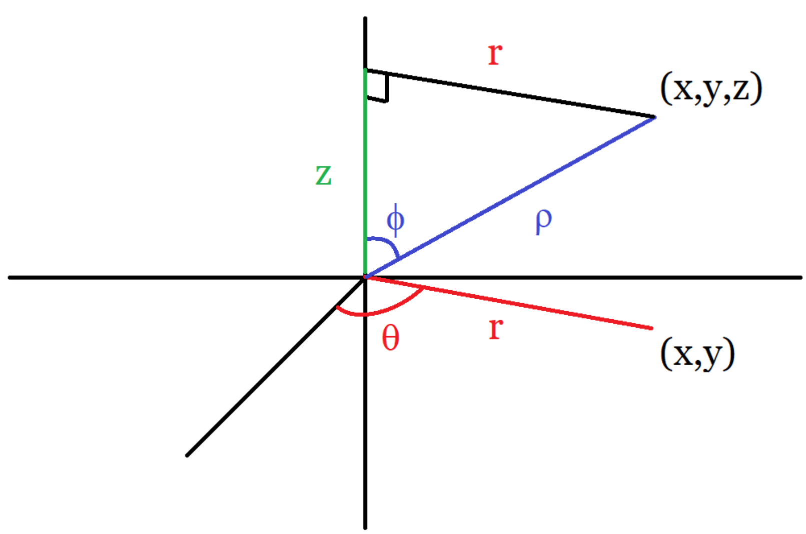
2. Derivation of Spherical Coordinates

Video: Derivation of Spherical Coordinates

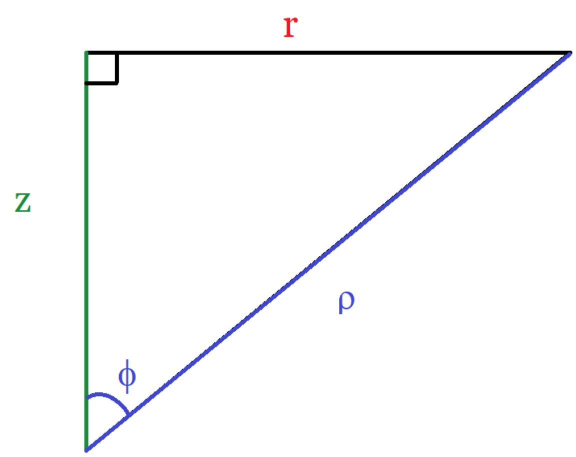
Goal: Find equations for x, y, z in terms of ρ, θ, ϕ (similar to $x = r\cos(\theta)$ for polar coordinates)

Note: You have to know how to derive this for the exams!

(1) **Picture:** Here r is the distance between O and (x, y) (like for cylindrical coordinates)



(2) Focus on the following triangle:



By SOHCAHTOA, we have:

$$\cos(\phi) = \frac{z}{\rho} \Rightarrow z = \rho\cos(\phi)$$

And also:

$$\sin(\phi) = \frac{r}{\rho} \Rightarrow r = \rho\sin(\phi)$$

(3) The rest is just polar coordinates and the formula for r above:

$$\begin{aligned} x =& r\cos(\theta) \Rightarrow x = \rho\sin(\phi)\cos(\theta) \\ y =& r\sin(\theta) \Rightarrow y = \rho\sin(\phi)\sin(\theta) \end{aligned}$$

Summary:

$$\begin{aligned} x &= \rho \sin(\phi)\cos(\theta) \\ y &= \rho \sin(\phi)\sin(\theta) \\ z &= \rho \cos(\phi) \end{aligned}$$

Note: Do not memorize this. On the exam, I will give you the equations for spherical coordinates.

3. Integrals with Spherical Coordinates

Now let's see why spherical coordinates are awesome! They allow us to simplify complicated integrals like crazy (= Bazooka of math)

Note: Spherical coordinates are great for spheres and cones.

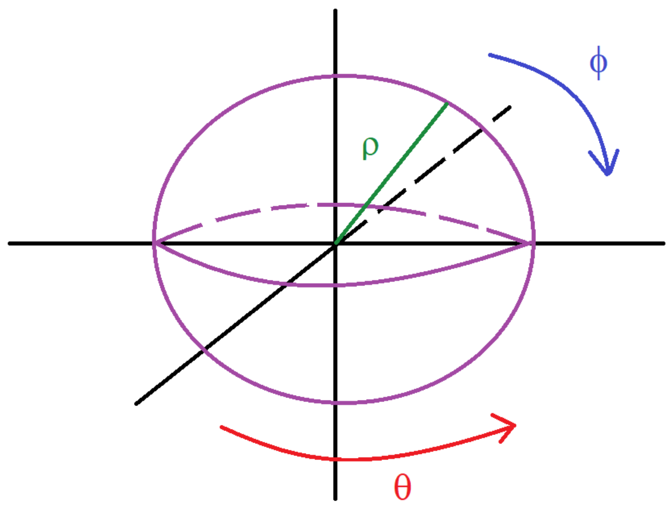
Example: Find the volume of a ball of radius R.

(1)

$$V = \int\int\int_E 1 \, dxdydz$$

$E =$ Ball of radius R

(2) **Picture:**



(3) **Inequalities:** Basically no restrictions on θ and ϕ

$$
\begin{aligned}
0 \leq \rho \leq R \\
0 \leq \theta \leq 2\pi \\
0 \leq \phi \leq \pi
\end{aligned}
$$

(4) **Integrate**

$$
\begin{aligned}
\int\int\int_E 1\, dxdydz &= \int_0^{\pi}\int_0^{2\pi}\int_0^{R} \rho^2 \sin(\phi) d\rho d\theta d\phi \\
&= \left(\int_0^R \rho^2 d\rho\right)\left(\int_0^{\pi} \sin(\phi) d\phi\right)\left(\int_0^{2\pi} 1 d\theta\right) \\
&= \frac{R^3}{3}(2)(2\pi) \\
&= \frac{4}{3}\pi R^3
\end{aligned}
$$

Note: Don't memorize the $\rho^2 \sin(\phi)$ term, it will be given to you on the exams

Note: *Very* roughly speaking, in polar coordinates we had $r dr d\theta$, but here we have:

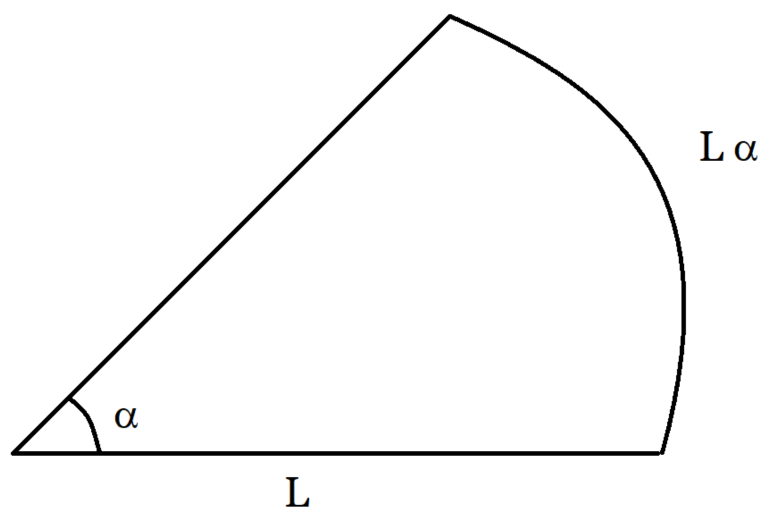
$$\rho\, r\, dr d\theta d\phi = \rho(\rho \sin(\phi))\, d\rho d\theta d\phi = \rho^2 \sin(\phi)\, d\rho d\theta d\phi$$

If you want a more geometric explanation, please see the *optional* appendix below:

4. Optional Appendix: $\rho^2 \sin(\phi)$

Question: Why do we get $\rho^2 \sin(\phi)$?

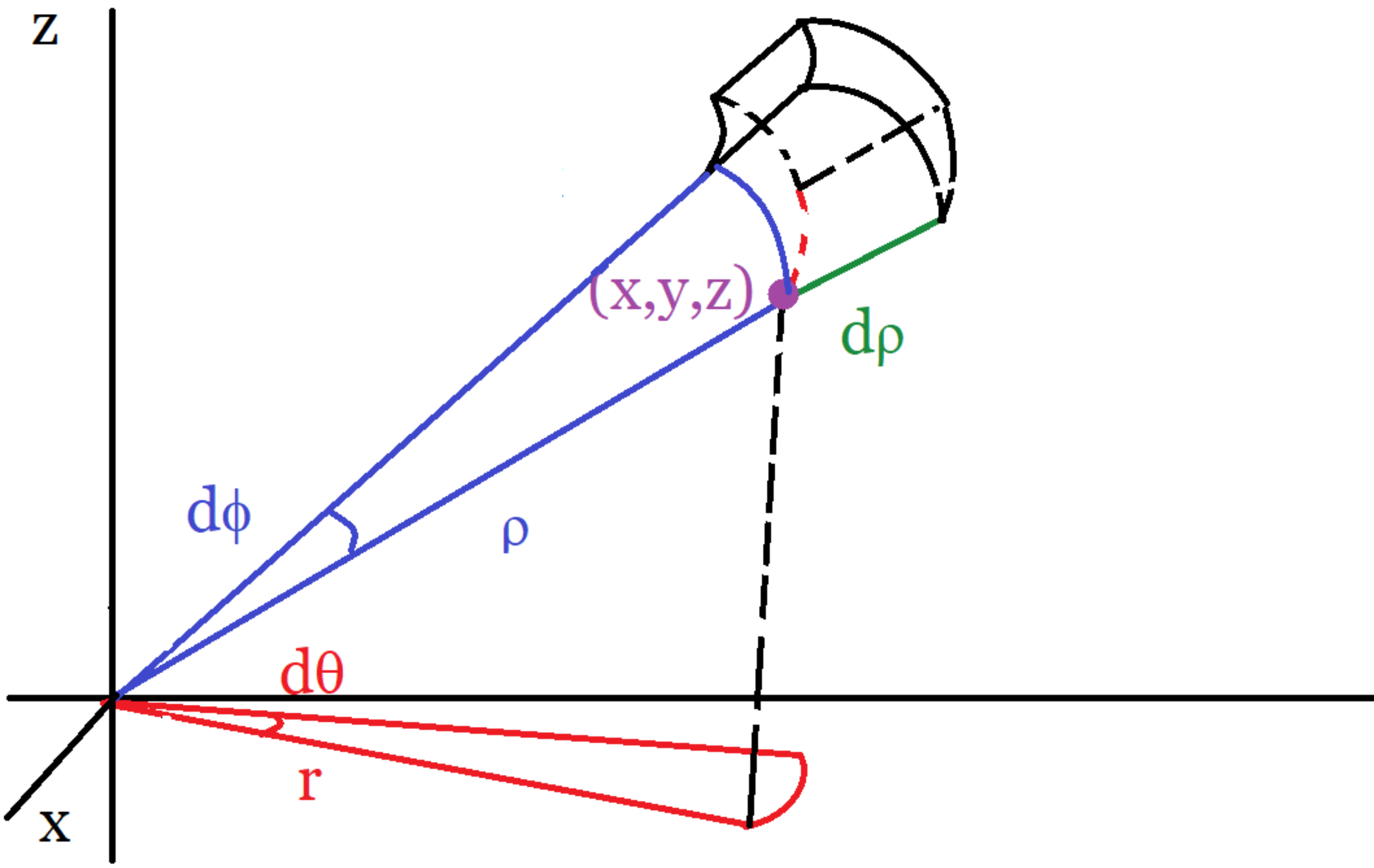
Recall: The length of an arc of radius L and angle α is $L\alpha$



This follows from proportionality: An angle of 2π (a full circle) corresponds to $2\pi L$, hence an angle of α corresponds to αL.

Now fix a point (x, y, z) and move around that point a little bit by changing ρ, θ, ϕ. If you do that, then in spherical coordinates you get a little wedge, as in the following picture:

Picture:

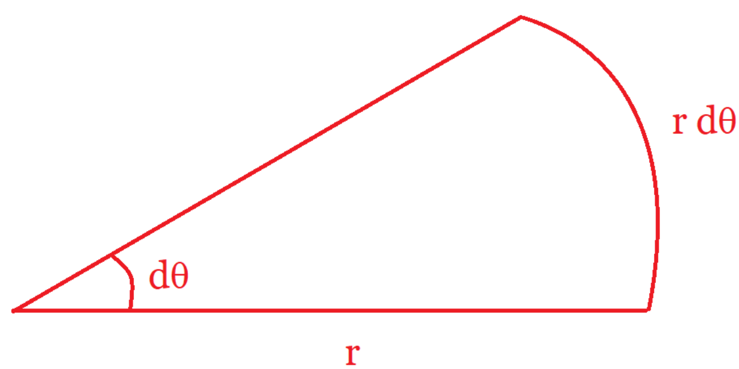


The volume of that wedge is approximately:

$$\text{Volume} \approx \text{Length} \times \text{Width} \times \text{Height}$$

(1) Length $= d\rho$ (Small change in the radius)

(2) Width $= rd\theta = \rho\sin(\phi)d\theta$



(3) Height $= \rho d\theta$ (because arclength of length ρ and angle $d\phi$)

Therefore:

$$\text{Volume} \approx (d\rho)(\rho\sin(\phi)d\theta)(\rho d\phi) = \rho^2\sin(\phi)d\rho d\theta d\phi$$