Today’s lecture is about spherical coordinates, which is the correct generalization of polar coordinates to three dimensions.

1. Spherical Coordinates

Idea: Represent points as \((\rho, \theta, \phi)\), where:

1. \(\rho = \text{distance from 0 to } (x, y, z)\) (RHOdius)
2. \(\theta = \text{angle between } (x, y) \text{ and } x\text{-axis}\) (THOrizontal)
3. \(\phi = \text{angle between } (x, y, z) \text{ and } z\text{-axis}\) (PHErtical)
Remarks:

(1) Think of $\theta$ as a longitude and $\phi$ as a latitude

(2) Constraints

$$\rho \geq 0$$
$$0 \leq \theta \leq 2\pi$$
$$0 \leq \phi \leq \pi$$
(3) Most important property: $x^2 + y^2 + z^2 = \rho^2$ (much easier!)

**Example:** Plot the following points:

(a) $(1, \frac{\pi}{4}, \frac{\pi}{6})$

Think like the hands of a robot. You move horizontally (= right) by $\frac{\pi}{4}$, starting from the $x$–axis and move vertically (= down) by $\frac{\pi}{6}$ starting from the $z$–axis.
(b) \((2, \frac{7\pi}{4}, \frac{\pi}{4})\)

(c) \((2, \frac{\pi}{3}, \frac{2\pi}{3})\)
Just like last time, this is useful because a lot of familiar objects can be written really neatly in terms of spherical coordinates.

**Example:** Sketch the following surfaces

(a) $\rho = 3$

\[ \sqrt{x^2 + y^2 + z^2} = 3 \Rightarrow x^2 + y^2 + z^2 = 9 \text{ Sphere} \]
Point: \( \rho = 3 \) is much easier than \( x^2 + y^2 + z^2 = 9 \)

(b) \( \theta = \frac{\pi}{3} \)

Half-plane through \( \theta = \frac{\pi}{3} \) (just like last time)
(c) $\phi = \frac{\pi}{6}$

Upper-cone!
Note: Lower cone is $\phi = \frac{5\pi}{6}$

2. Derivation of Spherical Coordinates

Video: Derivation of Spherical Coordinates

Goal: Find equations for $x, y, z$ in terms of $\rho, \theta, \phi$ (similar to $x = r \cos(\theta)$ for polar coordinates)

Note: You have to know how to derive this for the exams!

(1) Picture: Here $r$ is the distance between $O$ and $(x, y)$ (like for cylindrical coordinates)

(2) Focus on the following triangle:
By SOHCAHTOA, we have:

\[ \cos(\phi) = \frac{z}{\rho} \Rightarrow z = \rho \cos(\phi) \]

And also:

\[ \sin(\phi) = \frac{r}{\rho} \Rightarrow r = \rho \sin(\phi) \]

(3) The rest is just polar coordinates and the formula for \( r \) above:

\[
\begin{align*}
x &= r \cos(\theta) \Rightarrow x = \rho \sin(\phi) \cos(\theta) \\
y &= r \sin(\theta) \Rightarrow y = \rho \sin(\phi) \sin(\theta)
\end{align*}
\]

Summary:
\[ x = \rho \sin(\phi) \cos(\theta) \]
\[ y = \rho \sin(\phi) \sin(\theta) \]
\[ z = \rho \cos(\phi) \]

**Note:** Do not memorize this. On the exam, I will give you the equations for spherical coordinates.

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**3. Integrals with Spherical Coordinates**

Now let’s see why spherical coordinates are awesome! They allow us to simplify complicated integrals like crazy (= Bazooka of math)

**Note:** Spherical coordinates are great for spheres and cones.

**Example:** Find the volume of a ball of radius \( R \).

(1)

\[ V = \int \int \int_E 1\, dx\, dy\, dz \]

\[ E = \text{Ball of radius } R \]

(2) **Picture:**
(3) **Inequalities:** Basically no restrictions on $\theta$ and $\phi$

\[
0 \leq \rho \leq R \\
0 \leq \theta \leq 2\pi \\
0 \leq \phi \leq \pi
\]

(4) **Integrate**

\[
\int \int \int_E 1 \, dxdydz = \int_0^\pi \int_0^{2\pi} \int_0^R \rho^2 \sin(\phi) d\rho d\theta d\phi \\
= \left( \int_0^R \rho^2 d\rho \right) \left( \int_0^\pi \sin(\phi) d\phi \right) \left( \int_0^{2\pi} 1 d\theta \right) \\
= \frac{R^3}{3} (2)(2\pi) \\
= \frac{4}{3} \pi R^3
\]
**Note:** Don’t memorize the \( \rho^2 \sin(\phi) \) term, it will be given to you on the exams.

**Note:** Very roughly speaking, in polar coordinates we had \( r drd\theta \), but here we have:

\[
\rho r drd\theta d\phi = \rho (\rho \sin(\phi)) d\rho d\theta d\phi = \rho^2 \sin(\phi) d\rho d\theta d\phi
\]

If you want a more geometric explanation, please see the *optional* appendix below:

4. **Optional Appendix: \( \rho^2 \sin(\phi) \)**

**Question:** Why do we get \( \rho^2 \sin(\phi) \) ?

**Recall:** The length of an arc of radius \( L \) and angle \( \alpha \) is \( L\alpha \).
This follows from proportionality: An angle of $2\pi$ (a full circle) corresponds to $2\pi L$, hence an angle of $\alpha$ corresponds to $\alpha L$.

Now fix a point $(x, y, z)$ and move around that point a little bit by changing $\rho, \theta, \phi$. If you do that, then in spherical coordinates you get a little wedge, as in the following picture:

**Picture:**

![Diagram showing spherical coordinates and a small wedge](image-url)
The volume of that wedge is approximately:

\[
\text{Volume} \approx \text{Length} \times \text{Width} \times \text{Height}
\]

(1) \(\text{Length} = d\rho\) (Small change in the radius)

(2) \(\text{Width} = rd\theta = \rho \sin(\phi) d\theta\)

(3) \(\text{Height} = \rho d\theta\) (because arclength of length \(\rho\) and angle \(d\phi\))

Therefore:

\[
\text{Volume} \approx (d\rho)(\rho \sin(\phi) d\theta)(\rho d\phi) = \rho^2 \sin(\phi) d\rho d\theta d\phi
\]