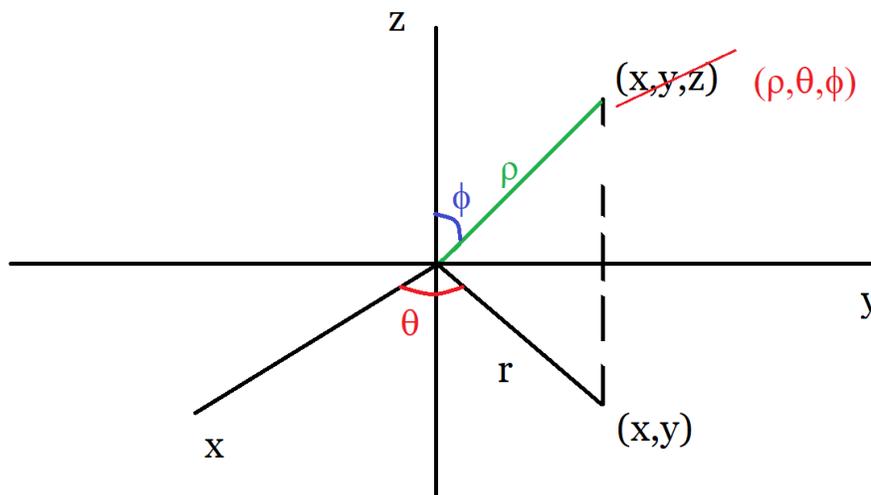


LECTURE 6: SPHERICAL COORDINATES (II)

1. QUICK RECAP

Previously, on *Dancing with the Spheres*, we discovered a new coordinate system:

(1) **Picture:**



(2) **Formulas:** (don't memorize)

Date: Tuesday, January 21, 2020.

$$x = \rho \sin(\phi) \cos(\theta)$$

$$y = \rho \sin(\phi) \sin(\theta)$$

$$z = \rho \cos(\phi)$$

(And $r = \rho \sin(\phi)$)

2. MASS OF THE SUN

Video: Mass of the Sun

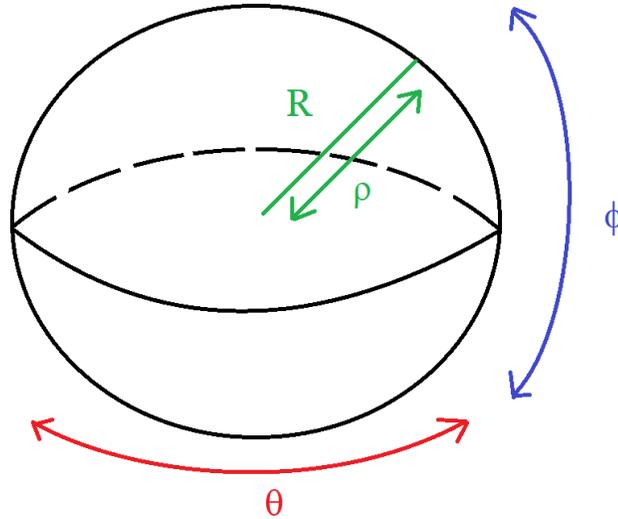
There is this meme that says: *In class: $2 + 2 = 4$. On the Exam: John has 4 apples, eats 1, calculate the mass of the sun.* So without further ado \dots let's calculate the mass of the sun!

Note: The mass of a solid E with density f is

$$\int \int \int_E f(x, y, z) \, dx dy dz$$

Example: Suppose the sun E is a ball of radius $R = 6.9 \times 10^{10}$ cm and density $f(x, y, z) = \sqrt{x^2 + y^2 + z^2} \, g/cm^3$. What is the mass of the sun?

(1) **Picture:**



(2) Inequalities:

$$0 \leq \rho \leq R$$

$$0 \leq \theta \leq 2\pi$$

$$0 \leq \phi \leq \pi$$

(3) Integrate:

$$\begin{aligned}
 \text{Mass} &= \int \int \int_E \sqrt{x^2 + y^2 + z^2} \, dx dy dz \\
 &= \int_0^\pi \int_0^{2\pi} \int_0^R \rho \rho^2 \sin(\phi) \, d\rho d\theta d\phi \quad (\text{Don't memorize}) \\
 &= 2\pi \left(\int_0^R \rho^3 \, d\rho \right) \left(\int_0^\pi \sin(\phi) \, d\phi \right) \\
 &= (2\pi) \frac{R^4}{4} (2) \\
 &= \pi R^4 \\
 &= 7.12 \times 10^{43} g
 \end{aligned}$$

Note: NASA uses the following density:

$$\frac{519}{R^4}\rho^4 - \frac{1630}{R^3}\rho^3 + \frac{1844}{R^2}\rho^2 - \frac{889}{R}\rho + 155$$

Which would give you 2.7×10^{33} grams. The actual mass of 1.98×10^{33} grams, so this is not bad at all!

3. INTEGRATION PRACTICE

Video: Spherical Coordinates and Ice Cream Cones

Example: $\int \int \int_E z \, dx \, dy \, dz$

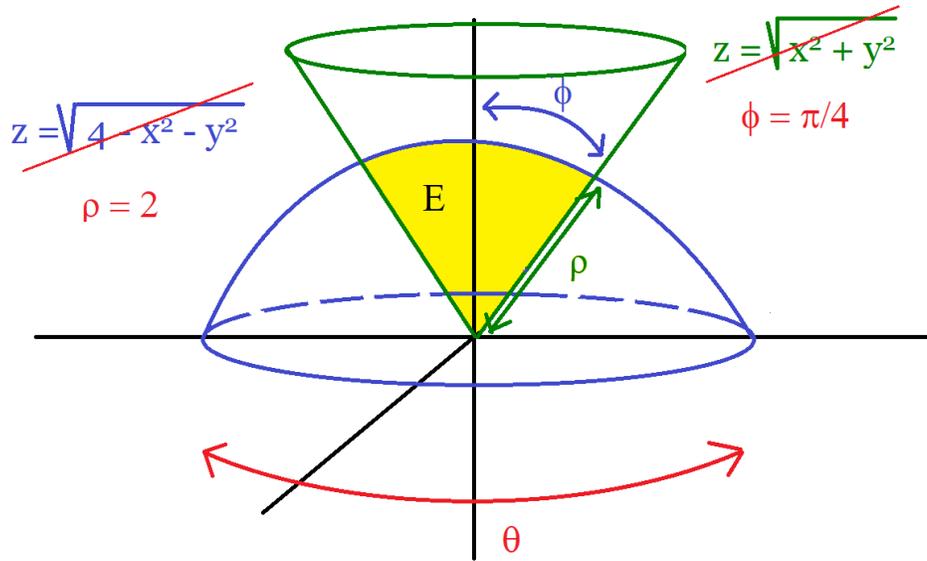
E is the region above $z = \sqrt{x^2 + y^2}$ and below $z = \sqrt{4 - x^2 - y^2}$

(1) **Picture:**

Note:

$$z = \sqrt{x^2 + y^2} \Rightarrow z^2 = x^2 + y^2 \text{ Cone}$$

$$z = \sqrt{4 - x^2 - y^2} \Rightarrow x^2 + y^2 + z^2 = 4 \text{ Sphere}$$



(2) Inequalities:

Note:

$$z = \sqrt{x^2 + y^2} = r \Rightarrow \rho \cos(\phi) = \rho \sin(\phi) \Rightarrow \cos(\phi) = \sin(\phi) \Rightarrow \phi = \frac{\pi}{4}$$

$$0 \leq \rho \leq 2$$

$$0 \leq \theta \leq 2\pi$$

$$0 \leq \phi \leq \frac{\pi}{4}$$

(3) Integrate:

$$\begin{aligned}
\int \int \int_E z \, dx \, dy \, dz &= \int_0^{\frac{\pi}{4}} \int_0^{2\pi} \int_0^2 \rho \cos(\phi) \rho^2 \sin(\phi) \, d\rho \, d\theta \, d\phi \\
&= 2\pi \left(\int_0^2 \rho^3 \, d\rho \right) \left(\int_0^{\frac{\pi}{4}} \sin(\phi) \cos(\phi) \, d\phi \right) \\
&= 2\pi \left(\frac{2^4}{4} \right) \left(\int_0^{\frac{1}{\sqrt{2}}} u \, du \right) \\
&\quad \text{Here we used } u = \sin(\phi), \, du = \cos(\phi) \, d\phi \\
&= \dots \\
&= 2\pi
\end{aligned}$$

Example: $\int \int \int_E \sqrt{x^2 + y^2} \, dx \, dy \, dz$

E : solid between $x^2 + y^2 + z^2 = 1$, $x^2 + y^2 + z^2 = 4$, $\phi = \frac{\pi}{6}$, $\phi = \frac{\pi}{3}$, ($z \geq 0$)

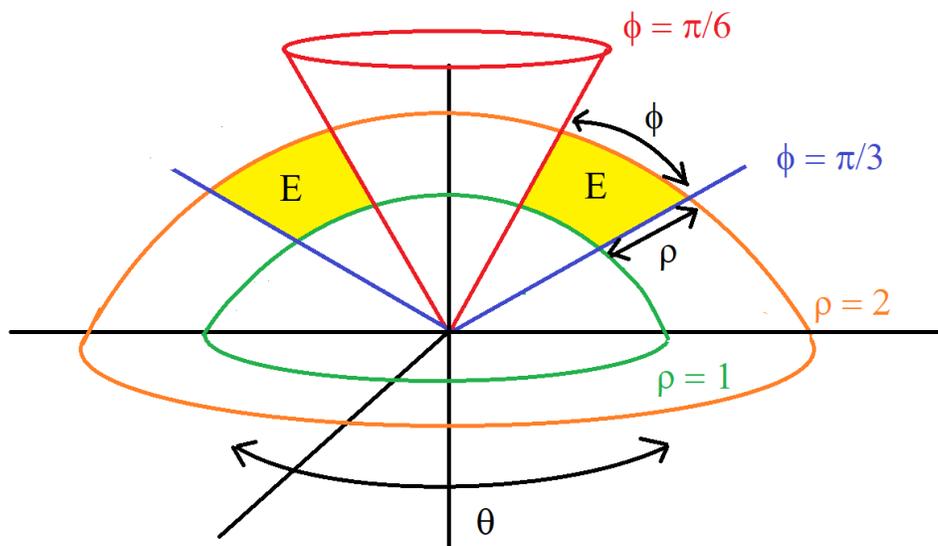
(1) **Picture:**

Note:

$$x^2 + y^2 + z^2 = 1 \Rightarrow \rho = 1$$

$$x^2 + y^2 + z^2 = 4 \Rightarrow \rho^2 = 4 \Rightarrow \rho = 2$$

$\phi = \frac{\pi}{6}$ and $\phi = \frac{\pi}{3}$ are cones.



Note: The picture above is a profile view; technically E should be rotated about the z -axis (like a solid of revolution)

(2) **Inequalities:**

$$\begin{aligned}
 1 &\leq \rho \leq 2 \\
 0 &\leq \theta \leq 2\pi \\
 \frac{\pi}{6} &\leq \phi \leq \frac{\pi}{3}
 \end{aligned}$$

(3) **Integrate:**

$$\begin{aligned}
\int \int \int_E \sqrt{x^2 + y^2} \, dx \, dy \, dz &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \int_0^{2\pi} \int_1^2 \rho \sin(\phi) \rho^2 \sin(\phi) \, d\rho \, d\theta \, d\phi \\
&= 2\pi \left(\int_1^2 \rho^3 \, d\rho \right) \left(\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin^2(\phi) \, d\phi \right) \\
&= 2\pi \left[\frac{\rho^4}{4} \right]_1^2 \left(\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{2} - \frac{1}{2} \cos(2\phi) \, d\phi \right) \\
&= 2\pi \left(\frac{15}{4} \right) \left[\frac{\phi}{2} - \frac{1}{4} \sin(2\phi) \right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} \\
&= \dots \\
&= \frac{5\pi^2}{8}
\end{aligned}$$

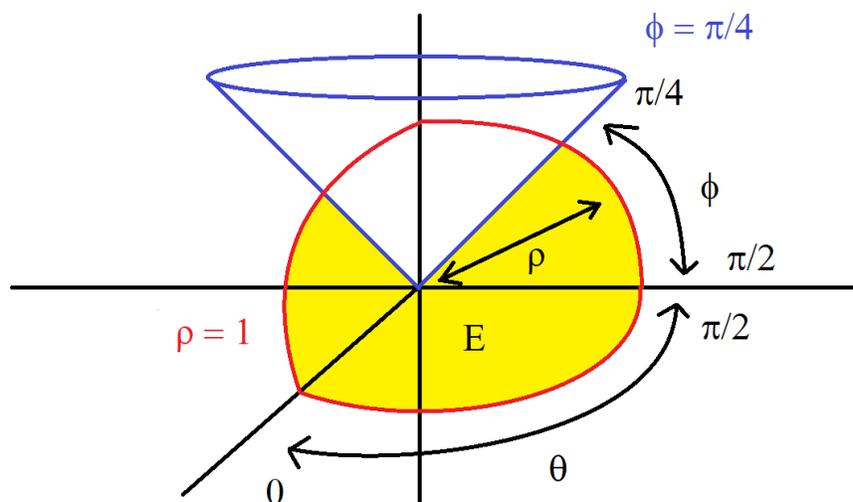
Example $\int \int \int_E x \, dx \, dy \, dz$, E : Solid under the cone $z = \sqrt{x^2 + y^2}$ and inside the sphere $x^2 + y^2 + z^2 = 1$, in the first octant.

(1) **Picture:**

Note:

$$z = \sqrt{x^2 + y^2} \Rightarrow \phi = \frac{\pi}{4}$$

$$x^2 + y^2 + z^2 = 1 \Rightarrow \rho = 1$$



(2) Inequalities:

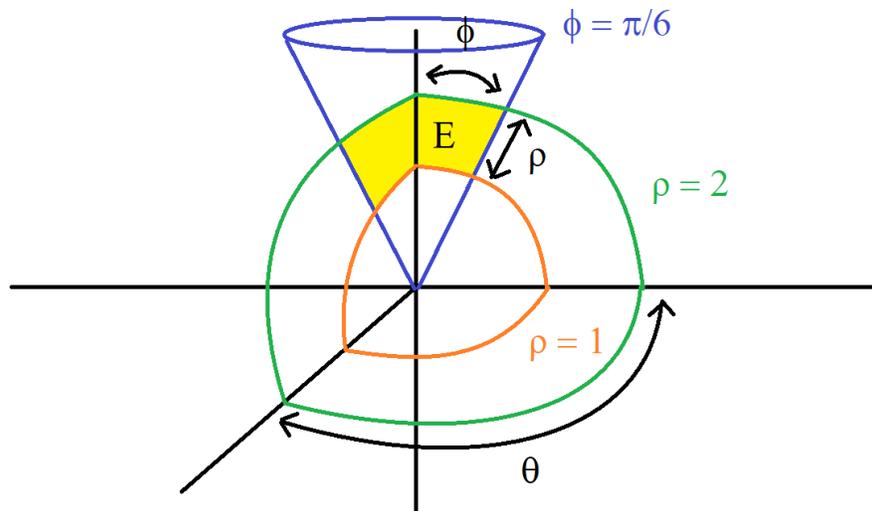
$$\begin{aligned} 0 &\leq \rho \leq 1 \\ 0 &\leq \theta \leq \frac{\pi}{2} \\ \frac{\pi}{4} &\leq \phi \leq \frac{\pi}{2} \end{aligned}$$

(3) Integrate:

$$\begin{aligned}
\int \int \int_E x \, dx \, dy \, dz &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^1 \rho \sin(\phi) \cos(\theta) \rho^2 \sin(\phi) \, d\rho \, d\theta \, d\phi \\
&= \left(\int_0^1 \rho^3 \, d\rho \right) \left(\int_0^{\frac{\pi}{2}} \cos(\theta) \, d\theta \right) \left(\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin^2(\phi) \, d\phi \right) \\
&= \dots \\
&= \left(\frac{1}{4} \right) (1) \left(\frac{\pi}{8} + \frac{1}{4} \right) \quad (\text{Similar to previous example}) \\
&= \frac{\pi}{32} + \frac{1}{16}
\end{aligned}$$

Extra Practice: (If time permits; Fall 2018 Midterm Question)
 $\int \int \int_E y \, dx \, dy \, dz$, E : Between $\rho = 1$, $\rho = 2$, above $\phi = \frac{\pi}{6}$, in first octant.

(1) **Picture:**



(2) **Inequalities:**

$$\begin{aligned}1 &\leq \rho \leq 2 \\0 &\leq \theta \leq \frac{\pi}{2} \\0 &\leq \phi \leq \frac{\pi}{6}\end{aligned}$$

(3) **Integrate:**

$$\begin{aligned}\int \int \int_E y \, dx dy dz &= \int_0^{\frac{\pi}{6}} \int_0^{\frac{\pi}{2}} \int_1^2 \rho \sin(\phi) \sin(\theta) \rho^2 \sin(\phi) \, d\rho d\theta d\phi \\&= \left(\int_1^2 \rho^3 d\rho \right) \left(\int_0^{\frac{\pi}{2}} \sin(\theta) d\theta \right) \left(\int_0^{\frac{\pi}{6}} \sin^2(\phi) d\phi \right) \\&= \dots \\&= \left(\frac{15}{4} \right) (1) \left(\frac{\pi}{12} - \frac{\sqrt{3}}{8} \right) \\&= \frac{5\pi}{16} - \frac{15\sqrt{3}}{32}\end{aligned}$$