LECTURE 7: CHANGE OF VARIABLES (I)

Welcome to the one and only integration technique in this course: $u-$sub!

1. $u-$sub the 2E way

Let me remind you of how to do $u-$sub from Math 2B, but I'll show you how to do it the Math 2E way (completely normal if this seems weird to you).

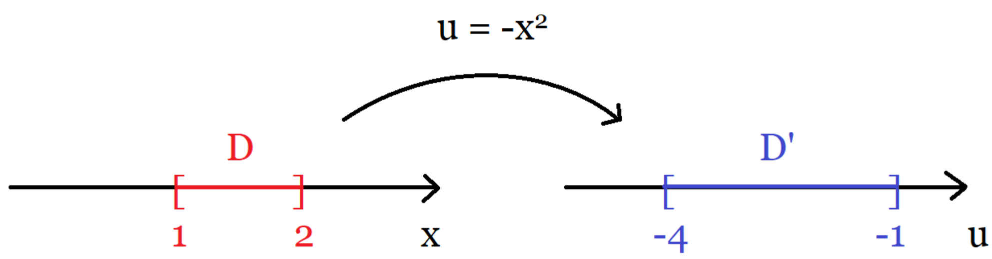
Example: Evaluate $\int_1^2 e^{-x^2}(-2x)dx$

(1) Let $u = -x^2$

(2) **Endpoints:** $u(1) = -1, u(2) = -4$.

So u turns $D = [1, 2]$ into $D' = \cancel{[-1, -4]} = [-4, -1]$.

Date: Wednesday, January 22, 2020.



(3) **du**: Beware of the absolute value! (makes sense, du should be positive)

$$du = \left|\frac{du}{dx}\right| dx = |-2x| dx = 2x dx \Rightarrow -2x dx = -du$$

(4) **Integrate**

$$\begin{aligned}\int_1^2 e^{-x^2}(-2x)dx &= \int_{[1,2]} e^{-x^2}(-2x)dx \\ &= \int_D e^{-x^2}(-2x)dx \\ &= \int_{D'} e^u(-du) \\ &= -\int_{[-4,-1]} e^u du \\ &= -\int_{-4}^{-1} e^u du \\ &= e^{-4} - e^{-1}\end{aligned}$$

2. Multivariable Examples

Video: The Jacobian

The good news is that for double and triple integrals, the process is the exact same as above!

Example:

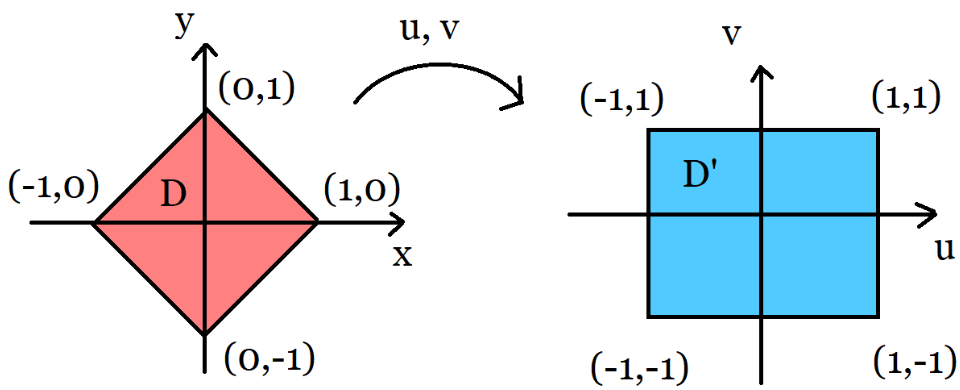
$$\int\int_D \sin\left(\frac{y-x}{y+x}\right) dxdy$$

Where D is the square with vertices $(-1,0),(0,-1),(1,0),(0,1)$.

(1) Let $u = y - x, v = y + x$

(2) **"Endpoints"**

Trick: Look at the values of u and v at the vertices:



$$\begin{aligned}(-1,0) \Rightarrow & x=-1, y=0 \\ \Rightarrow & u = y-x = 0-(-1) = 1, v = y+x = 0+(-1) = -1 \\ \Rightarrow & (1,-1)\end{aligned}$$

$$(0,-1) \Rightarrow u = -1-0 = -1, v = -1+0 = -1 \Rightarrow (-1,-1)$$

Similarly $(1,0)$ becomes $(-1,1)$ and $(0,1)$ becomes $(1,1)$.

So D' is a square with vertices $(1,-1),(-1,-1),(-1,1),(1,1)$

(3) "$du = \left|\frac{du}{dx}\right| dx$"

Here we get

$$dudv = \left|\frac{dudv}{dxdy}\right| dxdy$$

Problem: Before we only had one choice $\frac{du}{dx}$ but here we have many choices, like $\frac{\partial u}{\partial x}$ or $\frac{\partial v}{\partial y}$.

$$\frac{dudv}{dxdy} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = (-1)(1) - (1)(1) = -2$$

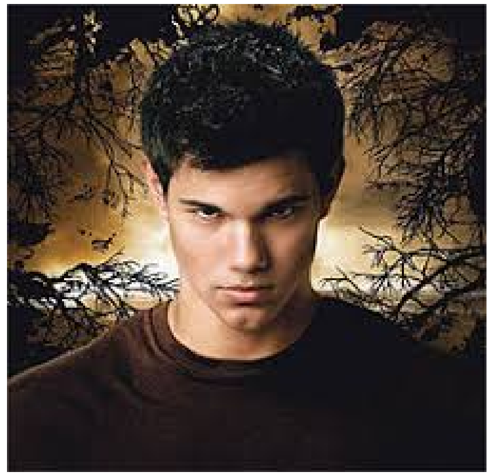
Therefore:

$$dudv = \left|-2\right| dxdy = 2dxdy \Rightarrow dxdy = \frac{1}{2} dudv$$

Remarks:

(a) This number (or its absolute value) is called the Jacobian, a tribute to Taylor Lautner in Twilight[1]

[1] Or maybe it's because of the mathematician Jacobi, I don't remember ☺



(b) Think of $\frac{dudv}{dxdy}$ as differentiating the hell out of everything: Differentiate all the functions u and v with respect to all the variables x and y

(c) Reminder:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

(d) We use the determinant because we ultimately want a number, not a matrix

(4) **Integrate:**

$$\begin{aligned}\int\int_D \sin\left(\frac{y-x}{y+x}\right) dxdy =& \int\int_{D'} \sin\left(\frac{u}{v}\right)\frac{1}{2}dudv\\ =&\frac{1}{2}\int_{-1}^{1}\int_{-1}^{1}\sin\left(\frac{u}{v}\right)dudv \quad \textit{Much} \text{ easier to integrate}\\ =&\cdots\\ =&0\end{aligned}$$

Note: Sometimes the change of variables is in the region D instead of the function:

Example:

$$\int\int_D xy\, dxdy$$

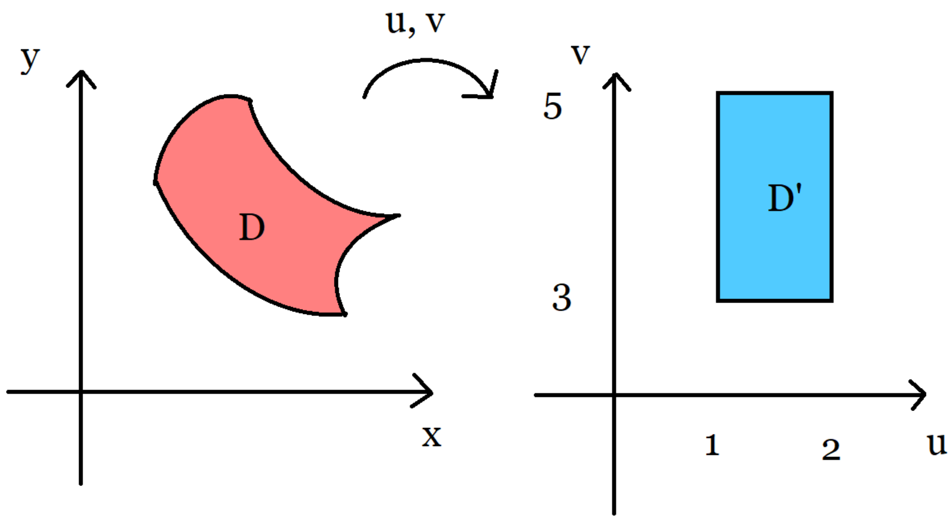
D is the region between $xy = 1, xy = 2, xy^2 = 3, xy^2 = 5$

(1) $u = xy, v = xy^2$

(2) **Find** D'

Note: You don't even need to know what D looks like!

$$\begin{aligned}1 \leq xy \leq 2 \Rightarrow& 1 \leq u \leq 2\\ 3 \leq xy^2 \leq 5 \Rightarrow& 3 \leq v \leq 5\end{aligned}$$



$$(3)\ dudv = \left|\frac{dudv}{dxdy}\right| dxdy,\ u = xy, v = xy^2$$

$$\frac{dudv}{dxdy} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} y & x \\ y^2 & 2xy \end{vmatrix} = y(2xy) - xy^2 = 2xy^2 - xy^2 = xy^2$$

$$dudv = \left|xy^2\right| dxdy = xy^2 dxdy = v dxdy \Rightarrow dxdy = \frac{1}{v} dudv$$

(4)

$$\begin{aligned}\int\int_D xydxdy &= \int\int_{D'} u\frac{1}{v}dudv \\ &= \int_3^5\int_1^2 \frac{u}{v}dudv \\ &= \cdots \\ &= \frac{3}{2}\ln\left(\frac{5}{3}\right)\end{aligned}$$

3. As easy as $\frac{4}{3}\pi abc$

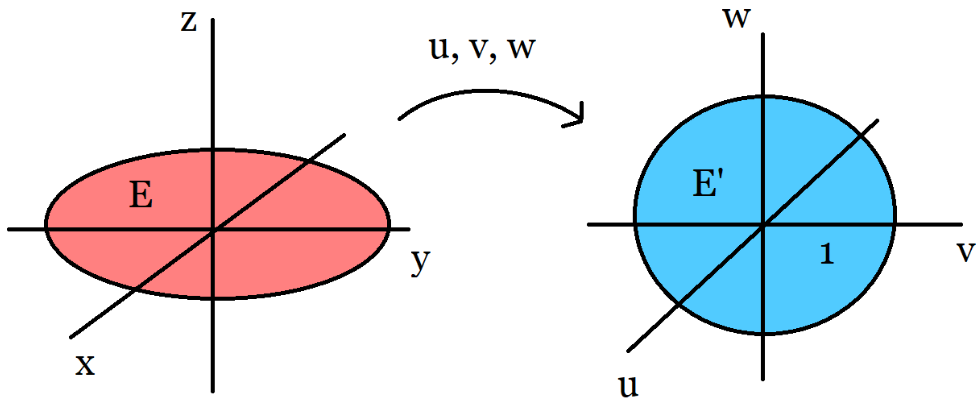
Example: Calculate the volume of the ellipsoid (here $a, b, c > 0$)

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 \leq 1$$

(1) $u = \frac{x}{a}, v = \frac{y}{b}, w = \frac{z}{c}$

(2) Find E':

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 \leq 1 \Rightarrow u^2 + v^2 + w^2 \leq 1 \text{ Ball of radius 1}$$



(3)

$$dudvdw = \left|\frac{dudvdw}{dxdydz}\right| dxdydz \ \left(u = \frac{x}{a}, v = \frac{y}{b}, w = \frac{z}{c}\right)$$

$$\frac{dudvdw}{dxdydz} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} \frac{1}{a} & 0 & 0 \\ 0 & \frac{1}{b} & 0 \\ 0 & 0 & \frac{1}{c} \end{vmatrix} = \frac{1}{abc}$$

$$dudvdw = \left|\frac{1}{abc}\right| dxdydz = \frac{1}{abc} dxdydz \Rightarrow dxdydz = abc\, dudvdw$$

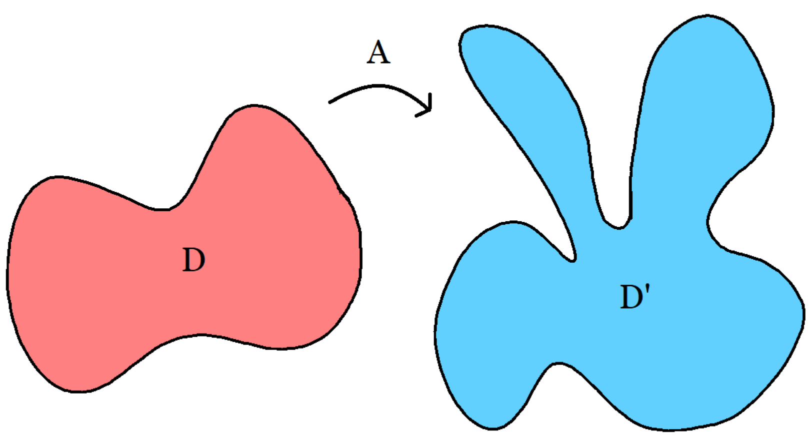
(4)

$$\begin{aligned}
Vol(E) &= \int\int\int_E 1\, dxdydz \\
&= \int\int\int_{E'} 1 abc\, dudvdw \\
&= abc \int\int\int_{E'} dudvdw \\
&= abc\, Vol(E') \\
&= abc\, \frac{4}{3}\pi 1^3 \quad (E' \text{ is a ball of radius } 1) \\
&= \frac{4}{3}\pi abc
\end{aligned}$$

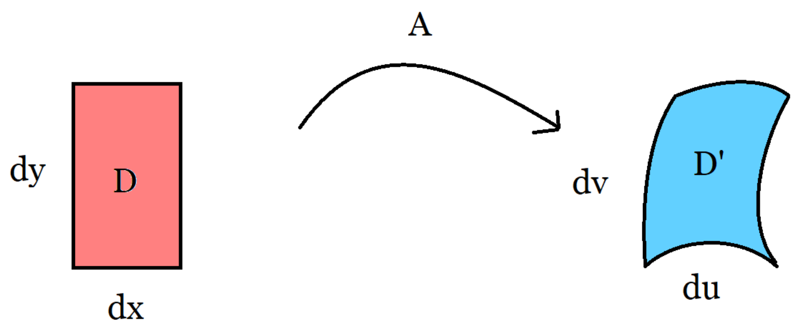
4. Optional Appendix: Why this works

Fact from Math 3A: If D and D' are regions and A is a matrix (= linear transformation) between them, then:

$$Area(D') = |\det(A)|\, Area(D)$$



Now suppose that D is a small rectangle with sides dx and dy. Then $A = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}$ transforms D into D', which is an object with sides du and dv:



On the one hand, the area of D' is approximately $dudv$, but on the other hand, by the formula above:

$$Area(D') = |\det A| \, Area(D)$$

$$dudv = \left| \det \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} \right| dxdy$$

$$dudv = \left| \frac{dudv}{dxdy} \right| dxdy$$

Finally, multiply both sides of the above by $f(u,v) = f(x,y)$ and integrate to get:

$$\int \int_{D'} f(u,v) dudv = \int \int_{D} f(x,y) \left| \frac{dudv}{dxdy} \right| dxdy$$