

## LECTURE 8: CHANGE OF VARIABLES (II)

Today we'll still do  $u$ -sub, but in reverse! Just like last time, let me motivate this again with a  $2B$  example:

### 1. MOTIVATION

**Video:** The Jacobian Part 2

**Example:**  $\int_0^1 \sqrt{1 - x^2} dx$

(1) Let  $x = \sin(u)$  (Notice here  $u$  is implicitly defined)

(2) **Endpoints:**

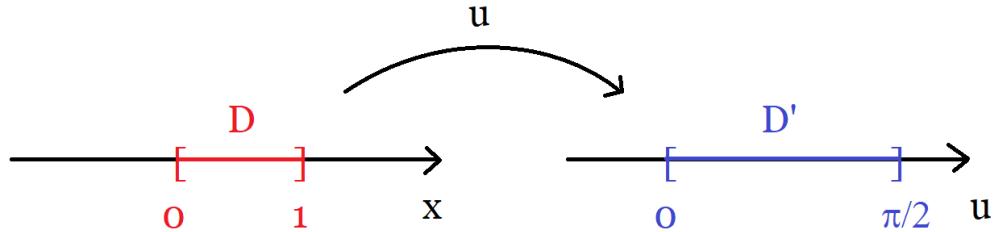
$$0 = \sin(u) \Rightarrow u = 0$$

$$1 = \sin(u) \Rightarrow u = \frac{\pi}{2}$$

So  $u$  turns  $D = [0, 1]$  into  $D' = [0, \frac{\pi}{2}]$

---

*Date:* Friday, January 24, 2020.



(3) This time you have  $dx$ , not  $du$

$$dx = \left| \frac{dx}{du} \right| du = |\cos(u)| du = \cos(u) du$$

(Here we used  $\cos(u) \geq 0$  since  $u$  is in  $[0, \frac{\pi}{2}]$ )

(4) Integrate

$$\begin{aligned} \int_{[0,1]} \sqrt{1 - x^2} dx &= \int_{[0, \frac{\pi}{2}]} \sqrt{1 - \sin^2(u)} \cos(u) du \\ &= \int_{[0, \frac{\pi}{2}]} \cos(u) \cos(u) du \\ &= \dots \\ &= \frac{\pi}{4} \end{aligned}$$

## 2. MULTIVARIABLE EXAMPLE

**Disclaimer:** This is a very stupid example, it's not something you would see in real life.

**Example:**

$$\int \int_D 4x + 8y \, dx dy$$

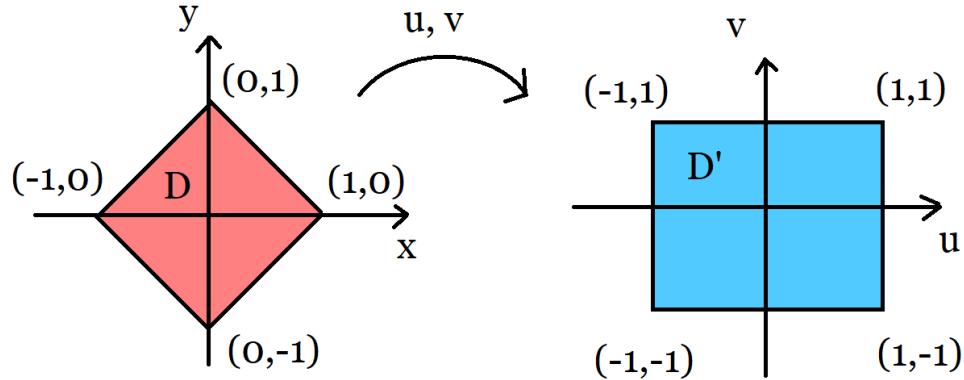
Where  $D$  is the square with vertices  $(-1, 0), (0, -1), (1, 0), (0, 1)$ .

- (1) Suppose someone tells you (**WILL** be given)

$$\begin{aligned} x &= \frac{1}{2}u + \frac{1}{2}v \\ y &= \frac{-1}{2}u + \frac{1}{2}v \end{aligned}$$

- (2) **Find**  $D'$

**Trick:** Look at the values of  $u$  and  $v$  at the vertices:



$$\begin{aligned}
 (1, 0) &\rightarrow x = 1, y = 0 \\
 \rightarrow 1 &= \frac{1}{2}u + \frac{1}{2}v \\
 0 &= \frac{-1}{2}u + \frac{1}{2}v \\
 \rightarrow \text{Add: } 1 &= \frac{v}{2} + \frac{v}{2} = v \\
 \rightarrow \text{Subtract: } 1 &= \frac{u}{2} + \frac{u}{2} = u \\
 \rightarrow u &= 1, v = 1 \\
 \rightarrow (1, 1)
 \end{aligned}$$

Similarly

$$\begin{aligned}
 (0, 1) &\rightarrow (-1, 1) \\
 (-1, 0) &\rightarrow (-1, -1) \\
 (0, -1) &\rightarrow (1, -1)
 \end{aligned}$$

So  $D'$  is a square with vertices  $(1, -1), (-1, -1), (-1, 1), (1, 1)$

(3) Jacobian

$$dxdy = \left| \frac{dxdy}{dudv} \right| dudv = \left| \frac{1}{2} \right| dudv = \frac{1}{2} dudv$$

$$\frac{dxdy}{dudv} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

(Here we used  $x = \frac{1}{2}u + \frac{1}{2}v, y = -\frac{1}{2}u + \frac{1}{2}v$ )

(4) Integrate:

$$\begin{aligned} \int \int_D 4x + 8y \, dxdy &= \int \int_{D'} \left[ 4\left(\frac{1}{2}u + \frac{1}{2}v\right) + 8\left(-\frac{1}{2}u + \frac{1}{2}v\right) \right] \frac{1}{2} dudv \\ &= \dots \\ &= \int_{-1}^1 \int_{-1}^1 -u + 3v \, dudv \\ &= 0 \end{aligned}$$

### 3. POLAR AND SPHERICAL COORDINATES

I know this was silly, but let me now tell you the real reason why this is so useful. Namely, it explains why we have  $r$  in  $rdrd\theta$ :

**Video:**  $rdrd\theta$

**Example:**

$$\int \int_D \tan^{-1}\left(\frac{y}{x}\right) \, dxdy$$

$D$  = Disk of Radius 1.

(1)

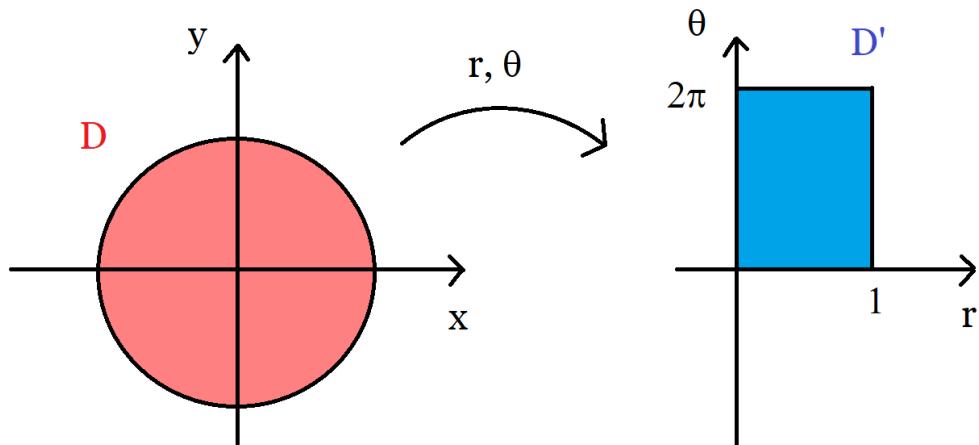
$$\begin{aligned}x &= r \cos(\theta) \\y &= r \sin(\theta)\end{aligned}$$

**Note:** Here  $r$  and  $\theta$  play the role of  $u$  and  $v$ .

(2) Find  $D'$

$$\begin{aligned}0 &\leq r \leq 1 \\0 &\leq \theta \leq 2\pi\end{aligned}$$

So  $r$  and  $\theta$  turn the disk  $D$  into a rectangle  $D'$



(3) Jacobian

$$dxdy = \left| \frac{dxdy}{drd\theta} \right| drd\theta = |r| drd\theta = \textcolor{red}{r} drd\theta$$

$$\frac{dxdy}{drd\theta} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos(\theta) & -r \sin(\theta) \\ \sin(\theta) & r \cos(\theta) \end{vmatrix} = r \cos^2(\theta) + r \sin^2(\theta) = \textcolor{red}{r}$$

(4) Integrate

$$\begin{aligned} \int \int_{\textcolor{red}{D}} \tan^{-1} \left( \frac{y}{x} \right) \textcolor{red}{dxdy} &= \int \int_{\textcolor{blue}{D'}} \theta \textcolor{blue}{r} drd\theta \\ &= \int_0^{2\pi} \int_0^1 \theta r dr d\theta \\ &= \pi^2 \end{aligned}$$

**Note:** Similarly, for cylindrical coordinates, we get  $dxdydz = r dr d\theta dz$

**Example:**

$$\int \int \int_E \frac{1}{\sqrt{x^2 + y^2 + z^2}} dx dy dz$$

$E$  : Ball of radius 1.

(1)

$$\begin{aligned} x &= \rho \sin(\phi) \cos(\theta) \\ y &= \rho \sin(\phi) \sin(\theta) \\ z &= \rho \cos(\phi) \end{aligned}$$

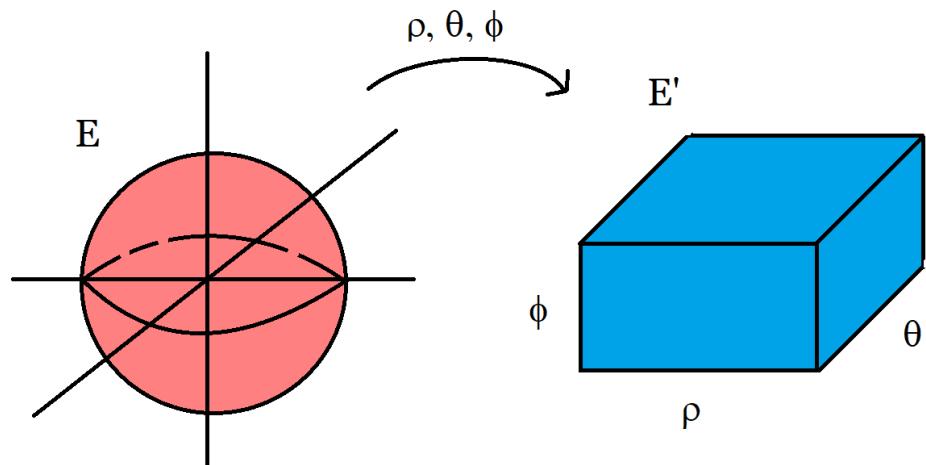
(2) Find  $E'$

$$0 \leq \rho \leq 1$$

$$0 \leq \theta \leq 2\pi$$

$$0 \leq \phi \leq \pi$$

So here  $E'$  is a box



(3) Jacobian

$$dxdydz = \left| \frac{dxdydz}{d\rho d\theta d\phi} \right| d\rho d\theta d\phi = \left| -\rho^2 \sin(\phi) \right| d\rho d\theta d\phi = \rho^2 \sin(\phi) d\rho d\theta d\phi$$

$$\begin{aligned}
 \frac{dxdydz}{d\rho d\theta d\phi} &= \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} \\
 &= \begin{vmatrix} \sin(\phi) \cos(\theta) & -\rho \sin(\phi) \sin(\theta) & \rho \cos(\phi) \cos(\theta) \\ \sin(\phi) \sin(\theta) & \rho \sin(\phi) \cos(\theta) & \rho \cos(\phi) \sin(\theta) \\ \cos(\phi) & 0 & -\rho \sin(\phi) \end{vmatrix} \\
 &= \dots \\
 &= -\rho^2 \sin(\phi)
 \end{aligned}$$

(Here we used  $x = \rho \sin(\phi) \cos(\theta)$ ,  $y = \rho \sin(\phi) \sin(\theta)$ ,  $z = \rho \cos(\phi)$ . Here it's best to expand the determinant along the third row. It's an annoying calculation, but know how to do this. See the AP1 Solutions in HW 3 for details)

(4) Integrate

$$\begin{aligned}
 \int \int \int_E \frac{1}{\sqrt{x^2 + y^2 + z^2}} dx dy dz &= \int \int \int_{E'} \frac{1}{\rho} \rho^2 \sin(\phi) d\rho d\theta d\phi \\
 &= \int_0^\pi \int_0^{2\pi} \int_0^1 \rho \sin(\phi) d\rho d\theta d\phi \\
 &= 2(2\pi) \left(\frac{1}{2}\right) \\
 &= 2\pi
 \end{aligned}$$

**Note:** At this point, I recommend checking out the last problem on the Fall 2018 midterm, which can also be found here:

## Hyperbolic Coordinates.

If you want to see a sick application of Jacobians, check out this completely optional video: [Jacobian 3](#)