LECTURE 9: VECTOR FIELDS

Welcome to the final chapter of your final calculus course! Like all final battles, this chapter is unbelievably hard, but also unbelievably exciting!

Today: A very gentle introduction to vector fields, just to show you how awesome they are!

1. Definition and Examples

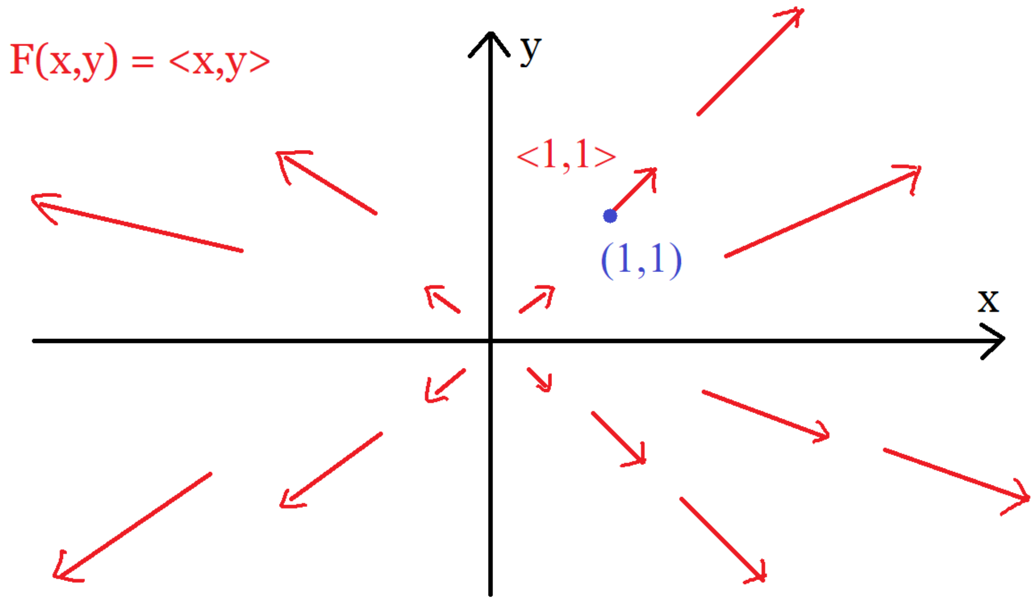
Definition: A **vector field** F is a function that associates to each point (x, y) a vector $F(x, y)$.

Really abstract definition, but a picture says 1000 words:

Example 1: Sketch $F(x, y) = \langle x, y \rangle$

$$F(1,1) = \langle 1,1 \rangle, F(3,1) = \langle 3,1 \rangle, F(2,2) = \langle 2,2 \rangle, F(-3,1) = \langle -3,1 \rangle$$

Date: Monday, January 27, 2020.



In the end, you get a bunch a vectors (one for each point), and that's precisely what a vector field is, just a collection of vectors!

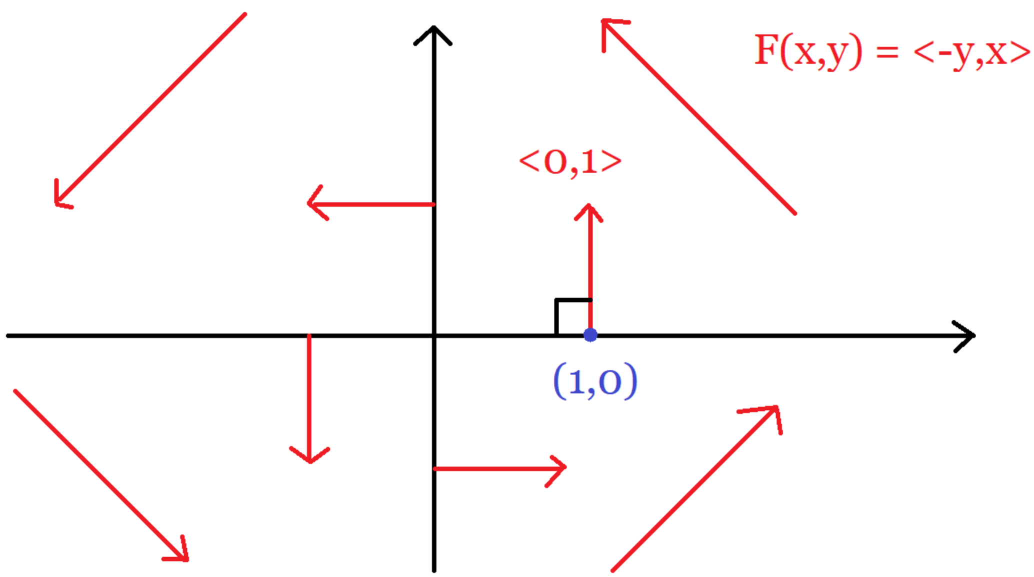
Many applications

- Force Field
- Velocity Field
- Gravitational Field
- Electrostatic Field
- Emotional Attraction?

(Whatever you like, there's probably a vector field for that)

Example 2: Sketch $F(x, y) = \langle -y, x \rangle$

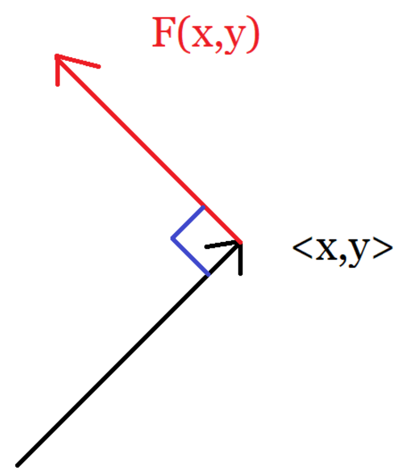
$$F(1,0) = \langle -0, 1 \rangle = \langle 0, 1 \rangle, F(0,1) = \langle -1, 0 \rangle, F(-1,0) = \langle 0, -1 \rangle, F(0,-1) = \langle 1, 0 \rangle$$



Note: Indeed, we have

$$F(x, y) \cdot \langle x, y \rangle = \langle -y, x \rangle \cdot \langle x, y \rangle = -yx + xy = 0$$

So $F(x, y) \perp \langle x, y \rangle$



Alternate notation:

$$F(x,y) = \langle -y, x\rangle = -y\mathbf{i} + x\mathbf{j}$$

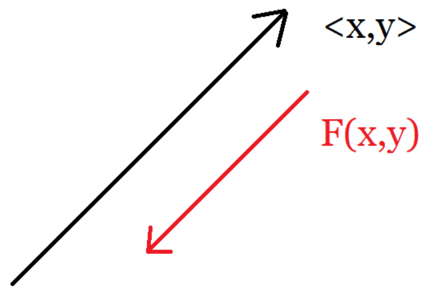
Where $\mathbf{i} = \langle 1, 0\rangle$ and $\mathbf{j} = \langle 0, 1\rangle$

Example 3: Draw:

$$F(x,y) = \left(-\frac{x}{\sqrt{x^2+y^2}}\right)\mathbf{i} + \left(-\frac{y}{\sqrt{x^2+y^2}}\right)\mathbf{j} = \left\langle -\frac{x}{\sqrt{x^2+y^2}}, -\frac{y}{\sqrt{x^2+y^2}}\right\rangle$$

Notice:

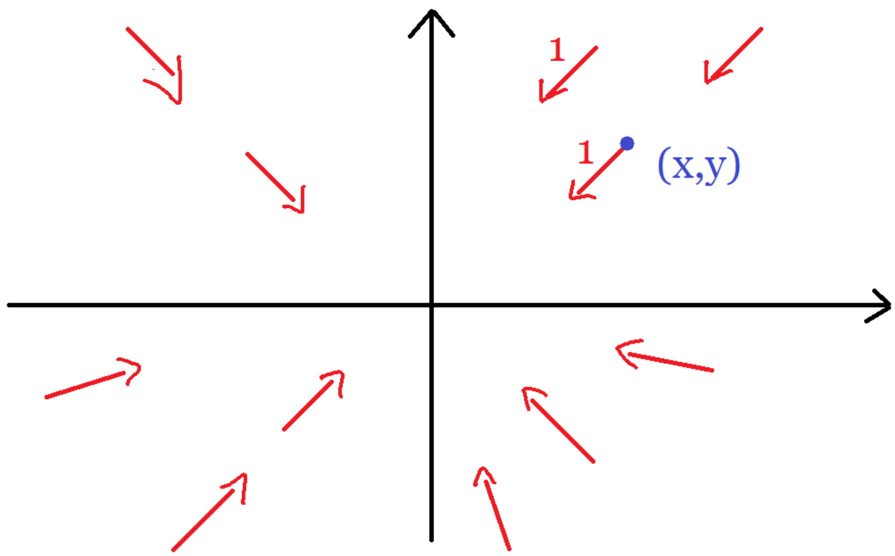
(1) $F(x,y) = \left(-\frac{1}{\sqrt{x^2+y^2}}\right)\langle x, y\rangle$, hence it points the opposite direction from $\langle x, y\rangle$



(2)

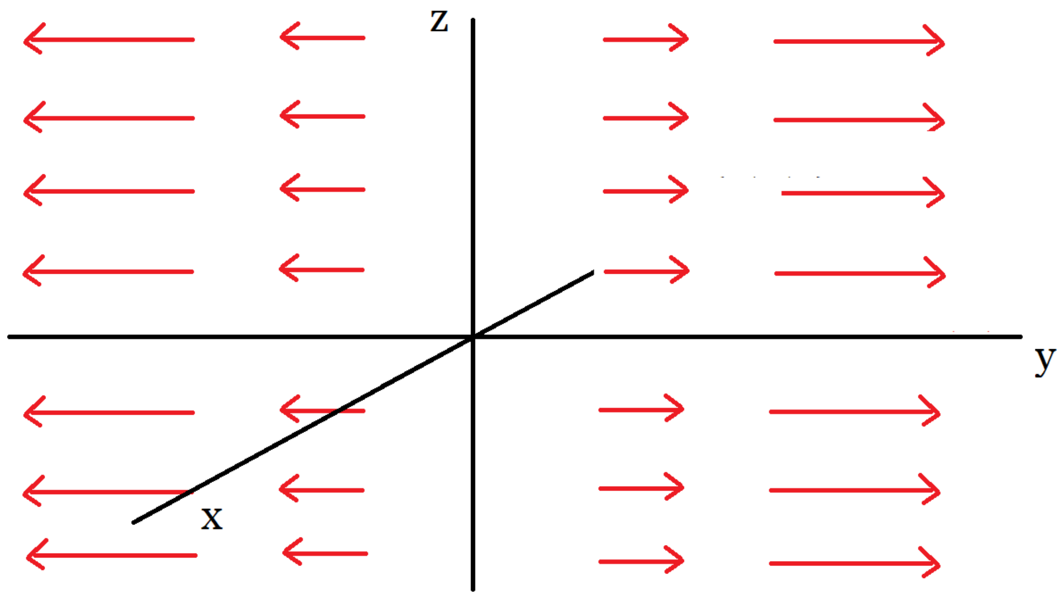
$$|F(x,y)| = \sqrt{\left(-\frac{x}{\sqrt{x^2+y^2}}\right)^2 + \left(-\frac{y}{\sqrt{x^2+y^2}}\right)^2} = \sqrt{\frac{x^2+y^2}{x^2+y^2}} = \sqrt{1} = 1$$

Hence $F(x,y)$ has length 1



Kind of like gravity/black hole!

Example 4: $F(x,y,z) = \langle 0, y, 0\rangle$ (independent of x and z, and proportional to $\langle 0,1,0\rangle$)



Point: Everything you learn in 2 dimensions can be generalized to 3 dimensions and beyond!

2. Gradient Fields

It turns out that there is an easy way of generating lots of nice vector fields, called **gradient fields**.

Definition: If $f(x, y)$ is a function, then

$$F = \nabla f = \langle f_x, f_y \rangle$$

is called the **gradient field** of f.

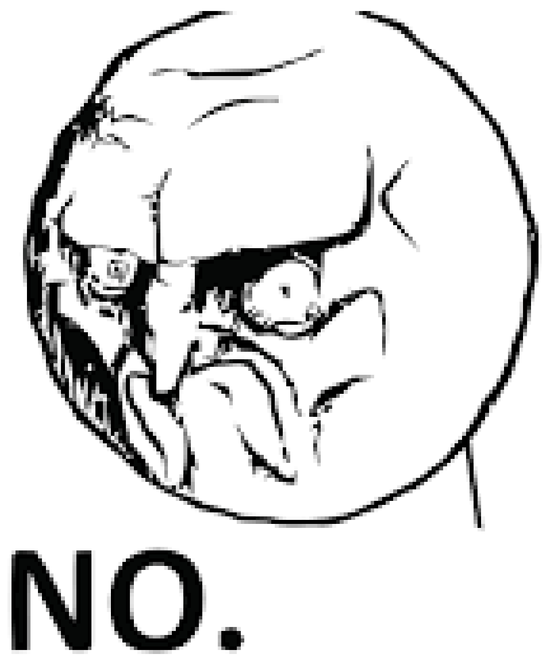
Example: The gradient field of $f(x, y) = x^2y - y^3$ is:

$$F = \nabla f = \left\langle (x^2y - y^3)_x, (x^2y - y^3)_y \right\rangle = \left\langle 2xy, x^2 - 3y^2 \right\rangle$$

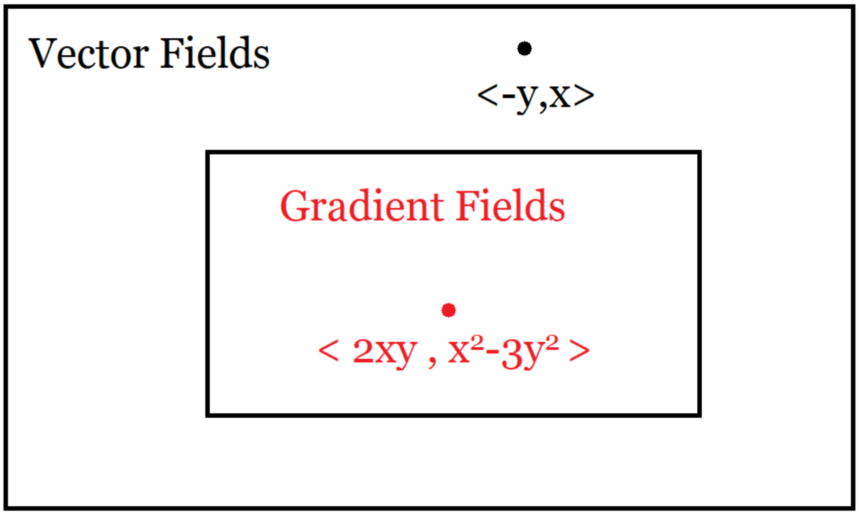
Notice that F is indeed a vector field! Nice vector field associated to f.

BIG QUESTION: Are all vector fields F gradient fields? (that is of the form ∇f for some f?)

Answer:



Example: (see 16.3) $F(x, y) = \langle -y, x \rangle$ (rotation field), F cannot be written as ∇f for any f.



But **IF** F is a gradient field, we call this ~~nice~~ conservative

Definition: F is **conservative** if $F = \nabla f$ for some f

Note: This sort of says F has an antiderivative[1]

Example: $F(x,y) = \left\langle 2xy, x^2 - 3y^2 \right\rangle$ is conservative because $F = \nabla f$ for $f(x,y) = x^2y - y^3$

Example:

$$F(x,y) = \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\rangle$$

is conservative, because if $f(x,y) = \sqrt{x^2+y^2}$, then:

[1]Conservative because of conservation of energy, not because of political preferences ☺

$$
\begin{aligned}
\nabla f =& \langle f_x, f_y \rangle \\
=& \left\langle \left(\sqrt{x^2+y^2}\right)_x, \left(\sqrt{x^2+y^2}\right)_y \right\rangle \\
=& \left\langle \frac{2x}{2\sqrt{x^2+y^2}}, \frac{2y}{2\sqrt{x^2+y^2}} \right\rangle \\
=& \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\rangle \\
=& F \qquad \checkmark
\end{aligned}
$$

GOALS OF THIS CHAPTER:

(1) What makes conservative vector fields so nice?

(2) How to determine F is conservative or not

3. Pretty Pictures

Check out this Reddit post for pretty examples of vector fields: Pretty Pictures