Math 2E – Suggested Homework 9

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Reading: Sections 16.5 (only the part about divergence) and 16.9. In Section 16.5, all we need for now is the definition of divergence, so you just need to know how to do Example 4. In section 16.9, ignore Example 3 and the discussion following it.

Important: Lecture on Friday, February 28 is cancelled. I will give you info about the make-up lecture soon. In any case, it will not affect this homework and/or quiz at all.

- **Section 16.5:** 1(b), 4(b), 7(b), 22, 25, 35 (Use $f = 1$), AP1
- **Section 16.9:** 5, 7, 8, 11, 12, 17, 18, 25, 29 (use 16.5.25), AP2, AP3, AP4

Note: Before tackling AP2 – AP4, I recommend looking over the notes for Lecture 22 (The Divergence Theorem II), which will be available sometime before Monday, March 2.

Additional Problem 1: Show that *any* function $f(x, y, z)$ can be written in the form $f = \text{div}(\mathbf{F})$ for some $\mathbf{F}$ (So there's no analog of conservative vector fields for div since *any* function is of this form)

Hint: Let $\mathbf{F} = \left\langle \int_0^x f(s, y, z)ds, 0, 0 \right\rangle$

Additional Problem 2: Suppose $f = f(x, y, z, t)$ satisfies the heat equation $f_t = \Delta f$ in E (here $\Delta f = f_{xx} + f_{yy} + f_{zz}$) and $\nabla f \cdot \mathbf{n} = 0$ on S. Define the mass of f at time t to be

$$M(t) = \iiint_E f(x,y,z,t)dxdydz$$

Show that $M'(t) = 0$ (so the mass of f is constant in time)

The next two problems deal with the formula

$$\boxed{Vol(E) = \iint_S \mathbf{F} \cdot d\mathbf{S}, \text{ where } \mathbf{F} = \frac{1}{3}\langle x,y,z\rangle}$$

Additional Problem 3: Let E be the ball centered at $(0,0,0)$ of radius R. Using the formula above and the fact that the volume of E is $\frac{4}{3}\pi R^3$, show that the surface area of a sphere centered at $(0,0,0)$ and radius R is $4\pi R^2$.

Hint: Use $\mathbf{F} \cdot d\mathbf{S} = \mathbf{F} \cdot \mathbf{n} dS$ and the fact that for the sphere S, we have $\mathbf{n} = \frac{1}{R}\langle x,y,z\rangle$

OMG Additional Problem 4: In this problem, we'll show that the surface area of a sphere is the derivative of the volume of the sphere!!! After this problem it shouldn't be surprising that $\left(\frac{4}{3}\pi r^3\right)' = 4\pi r^2$ or $(\pi r^2)' = 2\pi r$. (If you're interested in learning more about this, check out: Derivative of Volume is Surface Area)

(a) Let E be the ball centered at $(0,0,0)$ and radius R and let V be its volume. Using $V = \iiint_E 1\, dxdydz$ and the change of variables $u = \frac{x}{R}, v = \frac{y}{R}, z = \frac{z}{R}$, show that $V = CR^3$, where C is the volume of the ball centered at $(0,0,0)$ and radius 1.

(b) On the other hand, using the formula before AP3, show that $V = \left(\frac{R}{3}\right)S$, where S is the surface area of the sphere centered at $(0,0,0)$ and radius R.

(c) Use your answers from (a) and (b) to show that $S = 3CR^2$, and conclude that $V' = S$, so (in three dimensions) the derivative of the volume is the surface area.