Welcome to the first of four Fundamental Theorems of Calculus (FTC) in this course: The Fundamental Theorem of Line Integrals!

1. FTC for line integrals

**Recall:** (FTC, Math 2B)

\[ \int_a^b f'(x) \, dx = f(b) - f(a) = f(\text{end}) - f(\text{start}) \]

The multivariable analog of \( f'(x) \) is \( \nabla f(x) \), so we would like to say:

\[ \int_a^b \nabla f = f(b) - f(a) \]

But that doesn’t really make sense, since \( \nabla f \) is a vector! If only we could integrate a vector... but wait!

**Theorem:** FTC for Line Integrals For any curve \( C \):

\[ \int_C \nabla f \cdot dr = f(\text{end}) - f(\text{start}) = f(r(b)) - f(r(a)) \]
(This says: Integral of a derivative is \( f(b) - f(a) \))

**Example:** \( f(x, y) = x^3y + xy^3 \), \( C \) be any curve from \((0, 0)\) to \((1, 1)\)

Let

\[
F = \nabla f = \langle f_x, f_y \rangle = \langle 3x^2y + y^3, x^3 + 3xy^2 \rangle
\]

Then FTC says:
\[ \int_C F \cdot dr = \int_C \nabla f \cdot dr = f(1,1) - f(0,0) = [(1)^31 + 1(1)^3] - [(0)^30 + 0(0)^3] = 2 \]

So it’s easy to integrate \( \nabla f \)! In practice though, you do it in reverse:

**Example:** Let \( F(x,y) = \langle xy^2, x^2y \rangle \), \( C \) be any curve from \((1, 2)\) to \((3, 4)\). Find \( \int_C F \cdot dr \)

**Can show:** \( F = \nabla f \), where \( f(x,y) = \frac{1}{2}x^2y^2 \) (sort of like an antiderivative)

Then:

\[
\int_C F \cdot dr = \int_C \nabla f \cdot dr \\
= f(\text{end}) - f(\text{start}) \\
= f(3,4) - f(1,2) \\
= \frac{1}{2}(3)^2(4)^2 - \frac{1}{2}(1)^2(2)^2 \\
= 70
\]

Take-away: If \( F \) is nice/conservative (\( F = \nabla f \)), then \( \int_C F \cdot dr \) is easy to evaluate!

(And this precisely answers the question from 16.1 as to why conserv-ative vector fields are so nice!)

2. **Conservative Vector Fields**

**Problem:** How to determine if \( F \) is conservative?
It turns out that there is a really nice criterion for that!

**WARNING:** This trick only works in 2 dimensions! (will find a 3D analog of this in 16.5)

**2 dimensions:** Suppose

\[ F = \nabla f \]
\[ \langle P, Q \rangle = \langle f_x, f_y \rangle \]
\[ P = f_x \quad Q = f_y \]

**Recall:** (Clairaut/Schwarz)

\[ f_{xy} = f_{yx} \]
\[ (f_x)_y = (f_y)_x \]
\[ P_y = Q_x \]

**Fact:** If \( F = \langle P, Q \rangle \) is conservative, then \( P_y = Q_x \)

**Mnemonic:** Peyam = Quixotic

**Example:** \( F = \langle -y, x \rangle \) (rotation field), is \( F \) conservative?

\[ P = -y, \quad Q = x \]
\[ P_y = -1, \quad Q_x = 1 \]
\[ P_y \neq Q_x \]

No

So \( F \) conservative \( \Rightarrow P_y = Q_x \).

**Question:** \( P_y = Q_x \Rightarrow F \) conservative? “Yes”
(Yes if the domain of $F$ has no holes, no otherwise)

**Important Fact:** (if no holes)

\[ F \text{ conservative } \iff P_y = Q_x \]

**Example:** Is $F = \langle 3 + 2xy, x^2 - 3y^2 \rangle$ conservative?

\[ P_y = (3 + 2xy)_y = 2x, \ Q_x = (x^2 - 3y^2)_x = 2x, \ P_y = Q_x, \text{ so yes.} \]

**Note:** Intuitively: Conservative means: it doesn’t rotate, Not conservative means: it rotates.
3. Finding antiderivatives

Suppose $F$ is conservative, how to find an antiderivative of $F$?

**Example:** $F = \langle 3 + 2xy, x^2 - 3y^2 \rangle$, find $f$ such that $F = \nabla f$.

1) Check $P_y = Q_x \checkmark$

2) $F = \nabla f \Rightarrow \langle 3 + 2xy, x^2 - 3y^2 \rangle = \langle f_x, f_y \rangle$

Hence

$$f_x(x, y) = 3 + 2xy \Rightarrow f(x, y) = \int 3 + 2xy \, dx = 3x + x^2y + \text{JUNK}$$

This is saying that $f$ has the terms $3x$ and $x^2y$ in it, with possibly other terms

$$f_y(x, y) = x^2 - 3y^2 \Rightarrow f(x, y) = \int x^2 - 3y^2 \, dy = x^2y - y^3 + \text{JUNK}$$
Now collect all the terms (notice $x^2y$ appears twice here, so don’t count it twice)

3) 

$$f(x, y) = x^2y + 3x - y^2$$

(There might be other possibilities, but we just need one antiderivative)

Example: Find $f$ such that

$$F(x, y, z) = \langle y^2, 2xy + e^{3z}, 3ye^{3z} \rangle = \nabla f = \langle f_x, f_y, f_z \rangle$$

1) Check $F$ conservative. See 16.5 ✓

2) 

$$f_x(x, y, z) = y^2 \Rightarrow f(x, y, z) = \int y^2\,dx = xy^2 + \text{JUNK}$$

$$f_y(x, y, z) = 2xy + e^{3z} \Rightarrow f(x, y, z) = \int 2xy + e^{3z}\,dy = xy^2 + ye^{3z} + \text{JUNK}$$

$$f_z(x, y, z) = 3ye^{3z} \Rightarrow f(x, y, z) = \int 3ye^{3z}\,dz = ye^{3z} + \text{JUNK}$$

3) Hence $f(x, y, z) = xy^2 + ye^{3z}$

4. Putting it all together

Video: FTC for Line Integrals
(Will do many more examples next time)

**Example:** \( \int_C F \cdot dr, \; F(x, y) = \langle x^2 y^3, x^3 y^2 \rangle \)

\( C \) : is the curve:

\[
\begin{aligned}
    x(t) &= \cos(t) \\
    y(t) &= 2 \sin(t) \\
    0 &\leq t \leq \frac{\pi}{2}
\end{aligned}
\]

**Note:** *Could* do it directly, but it becomes way harder (sometimes even impossible) to integrate

1. **Picture:**

   ![Diagram](image)

   \((0,2)\) \hspace{1cm} \((1,0)\)

2. **Check:**
\begin{align*}
P_y &= (x^2 y^3)_x = 3x^2 y^2 \\
Q_x &= (x^3 y^2)_y = 3x^2 y^2
\end{align*}

(3)

\[
F = \nabla f \Rightarrow \langle x^2 y^3, x^3 y^2 \rangle = \langle f_x, f_y \rangle
\]

\[f_x(x, y) = x^2 y^3 \Rightarrow f(x, y) = \int x^2 y^3 \, dx = \frac{1}{3} x^3 y^3 + \text{JUNK}
\]

\[f_y(x, y) = x^3 y^2 \Rightarrow f(x, y) = \int x^3 y^2 \, dy = \frac{1}{3} x^3 y^3 + \text{JUNK}
\]

\[
f(x, y) = \frac{1}{3} x^3 y^3
\]

(4)

\[
\int_C F \cdot dr = \int_C \nabla f \cdot dr \\
= f(\text{end}) - f(\text{start}) \\
= f(0, 2) - f(1, 0) \\
= \frac{1}{3} (0)^3 (2)^3 - \frac{1}{3} (1)^3 (0)^3 \\
= 0
\]
5. **Appendix: Proof of FTC**

Consider

\[
\int_{a}^{b} \frac{d}{dt}f(r(t))dt
\]

On the one hand, this equals

\[
\int_{a}^{b} \frac{d}{dt}f(r(t))dt = f(r(b)) - f(r(a))
\]

On the other hand, by the Chen Lu (Chain Rule):

\[
\frac{d}{dt}f(r(t)) = \frac{d}{dt}f(x(t), y(t)) = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} = (f_x)(x'(t)) + (f_y)(y'(t)) = \langle f_x, f_y \rangle \cdot \langle x'(t), y'(t) \rangle = \nabla f(x(t), y(t)) \cdot r'(t)
\]

Therefore:

\[
\int_{a}^{b} \frac{d}{dt}f(r(t))dt = \int_{a}^{b} \nabla f(r(t)) \cdot r'(t) = \int_{C} \nabla f \cdot dr
\]

Combining the two, we get:

\[
\int_{C} \nabla f \cdot dr = \int_{a}^{b} \frac{d}{dt}f(r(t))dt = f(r(b)) - f(r(a))
\]