LECTURE 12: FTC FOR LINE INTEGRALS (I)

Welcome to the first of four Fundamental Theorems of Calculus (FTC) in this course: The Fundamental Theorem of Line Integrals!

1. FTC for line integrals

Recall: (FTC, Math 2B)

$$\int_a^b f'(x)dx = f(b) - f(a) = f(end) - f(start)$$

The multivariable analog of $f'(x)$ is $\nabla f(x)$, so we would like to say:

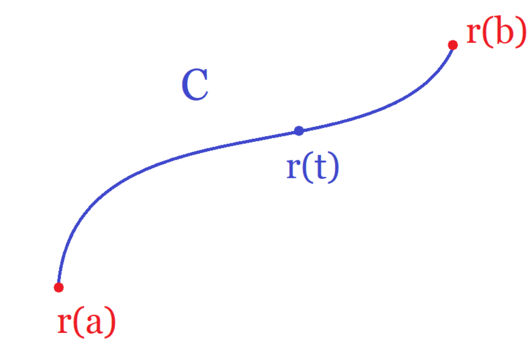
$$\int_a^b \nabla f = f(b) - f(a)$$

But that doesn't really make sense, since ∇f is a vector! If only we could integrate a vector... but wait!

Theorem: FTC for Line Integrals For any curve C:

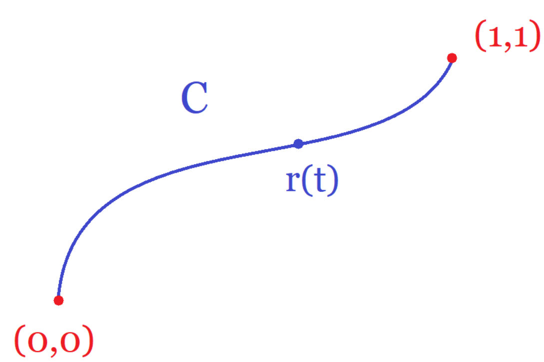
$$\int_C \nabla f \cdot dr = f(end) - f(start) = f(r(b)) - f(r(a))$$

Date: Monday, February 3, 2020.



(This says: Integral of a derivative is $f(b) - f(a)$)

Example: $f(x, y) = x^3 y + x y^3$, C be any curve from $(0, 0)$ to $(1, 1)$



Let

$$F = \nabla f = \langle f_x, f_y \rangle = \left\langle 3x^2 y + y^3, x^3 + 3xy^2 \right\rangle$$

Then FTC says:

$$\int_C F \cdot dr = \int_C \nabla f \cdot dr = f(1,1) - f(0,0) = \left[(1)^3 1 + 1(1)^3\right] - \left[(0)^3 0 + 0(0)^3\right] = 2$$

So it's easy to integrate ∇f! In practice though, you do it in reverse:

Example: Let $F(x,y) = \langle xy^2, x^2y \rangle$, C be any curve from $(1,2)$ to $(3,4)$. Find $\int_C F \cdot dr$

Can show: $F = \nabla f$, where $f(x,y) = \frac{1}{2}x^2y^2$ (sort of like an antiderivative)

Then:

$$\begin{aligned}
\int_C F \cdot dr = \int_C \nabla f \cdot dr \\
&= f(end) - f(start) \\
&= f(3,4) - f(1,2) \\
&= \frac{1}{2}(3)^2(4)^2 - \frac{1}{2}(1)^2(2)^2 \\
&= 70
\end{aligned}$$

Take-away: If F is nice/conservative ($F = \nabla f$), then $\int_C F \cdot dr$ is <u>easy</u> to evaluate!

(And this precisely answers the question from 16.1 as to why conservative vector fields are so nice!)

2. Conservative Vector Fields

Problem: How to determine if F is conservative?

It turns out that there is a really nice criterion for that!

WARNING: This trick only works in 2 dimensions! (will find a 3D analog of this in 16.5)

2 dimensions: Suppose

$$\begin{aligned} F &= \nabla f \\ \langle P, Q \rangle &= \langle f_x, f_y \rangle \\ P = f_x & \quad Q = f_y \end{aligned}$$

Recall: (Clairaut/Schwarz)

$$\begin{aligned} f_{xy} &= f_{yx} \\ (f_x)_y &= (f_y)_x \\ P_y &= Q_x \end{aligned}$$

Fact: If $F = \langle P, Q \rangle$ is conservative, then $P_y = Q_x$

Mnemonic: Peyam = Quixotic

Example: $F = \langle -y, x \rangle$ (rotation field), is F conservative?

$$\begin{gathered} P = -y, \ Q = x \\ P_y = -1, \ Q_x = 1 \\ P_y \neq Q_x \\ \text{No} \end{gathered}$$

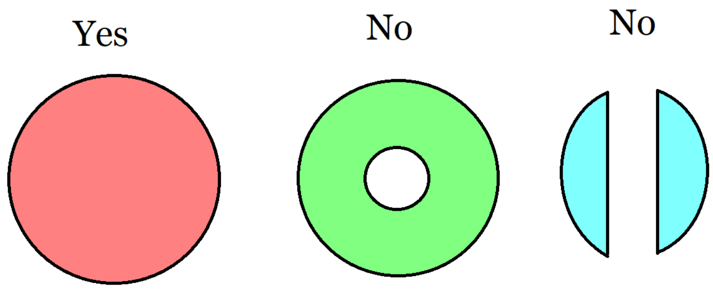
So F conservative $\Rightarrow P_y = Q_x$.

Question: $P_y = Q_x \Rightarrow F$ conservative? "Yes"

(Yes if the domain of F has no holes, no otherwise)

Important Fact: (if no holes)

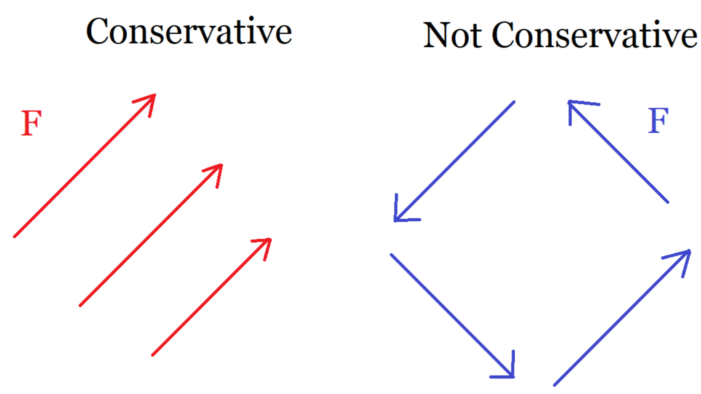
$$F \text{ conservative } \Leftrightarrow P_y = Q_x$$



Example: Is $F = \langle 3 + 2xy, x^2 - 3y^2 \rangle$ conservative?

$P_y = (3 + 2xy)_y = 2x$, $Q_x = (x^2 - 3y^2)_x = 2x$, $P_y = Q_x$, so yes.

Note: Intuitively: Conservative means: it doesn't rotate, Not conservative means: it rotates.



3. Finding antiderivatives

Suppose F is conservative, how to find an antiderivative of F?

Example: $F = \left\langle 3 + 2xy, x^2 - 3y^2 \right\rangle$, find f such that $F = \nabla f$.

1) Check $P_y = Q_x$ ✓

2) $F = \nabla f \Rightarrow \left\langle 3 + 2xy, x^2 - 3y^2 \right\rangle = \langle f_x, f_y \rangle$

Hence

$$f_x(x,y) = 3 + 2xy \Rightarrow f(x,y) = \int 3 + 2xy\, dx = 3x + x^2 y + \text{ JUNK}$$

This is saying that f has the terms $3x$ and $x^2 y$ in it, with possibly other terms

$$f_y(x,y) = x^2 - 3y^2 \Rightarrow f(x,y) = \int x^2 - 3y^2 dy = x^2 y - y^3 + \text{ JUNK}$$

Now collect all the terms (notice x^2y appears twice here, so don't count it twice)

3)

$$f(x,y) = x^2y + 3x - y^2$$

(There might be other possibilities, but we just need *one* antiderivative)

Example: Find f such that

$$F(x,y,z) = \left\langle y^2, 2xy + e^{3z}, 3ye^{3z} \right\rangle = \nabla f = \langle f_x, f_y, f_z \rangle$$

1) Check F conservative. See 16.5 ✓

2)

$$f_x(x,y,z) = y^2 \Rightarrow f(x,y,z) = \int y^2 dx = xy^2 + \text{ JUNK}$$

$$f_y(x,y,z) = 2xy + e^{3z} \Rightarrow f(x,y,z) = \int 2xy + e^{3z} dy = xy^2 + ye^{3z} + \text{ JUNK}$$

$$f_z(x,y,z) = 3ye^{3z} \Rightarrow f(x,y,z) = \int 3ye^{3z} dz = 3y\frac{e^{3z}}{3} = ye^{3z} + \text{ JUNK}$$

3) Hence $f(x,y,z) = xy^2 + ye^{3z}$

4. Putting it all together

Video: FTC for Line Integrals

(Will do many more examples next time)

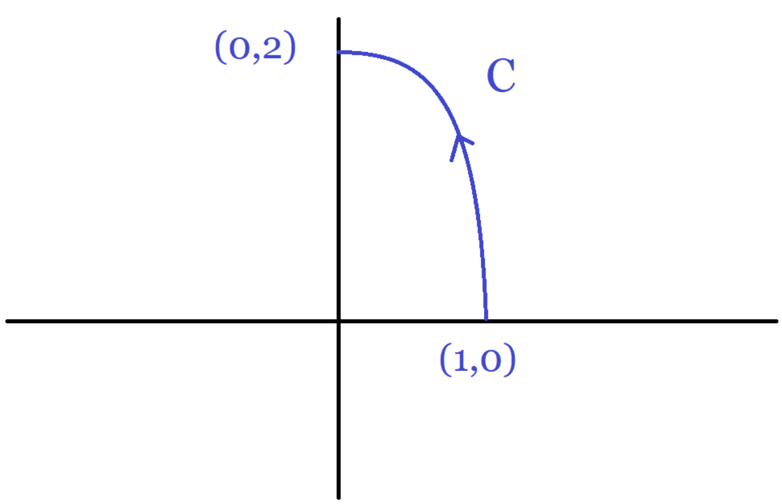
Example: $\int_C F \cdot dr$, $F(x, y) = \left\langle x^2y^3, x^3y^2 \right\rangle$

C : is the curve:

$$\begin{cases} x(t) = \cos(t) \\ y(t) = 2\sin(t) \\ 0 \leq t \leq \frac{\pi}{2} \end{cases}$$

Note: *Could* do it directly, but it becomes way harder (sometimes even impossible) to integrate

(1) **Picture:**



(2) **Check:**

$$\begin{aligned}P_y &= \left(x^2y^3\right)_x = 3x^2y^2\\ Q_x &= \left(x^3y^2\right)_y = 3x^2y^2\end{aligned}$$

(3)

$$F = \nabla f \Rightarrow \left\langle x^2y^3, x^3y^2\right\rangle = \langle f_x, f_y\rangle$$

$$f_x(x,y) = x^2y^3 \Rightarrow f(x,y) = \int x^2y^3dx = \frac{1}{3}x^3y^3 + \text{ JUNK}$$

$$f_y(x,y) = x^3y^2 \Rightarrow f(x,y) = \int x^3y^2dy = \frac{1}{3}x^3y^3 + \text{ JUNK}$$

$$f(x,y) = \frac{1}{3}x^3y^3$$

(4)

$$\begin{aligned}\int_C F\cdot dr =& \int_C \nabla f\cdot dr\\ =& f(end) - f(start)\\ =& f(0,2) - f(1,0)\\ =& \frac{1}{3}(0)^3(2)^3 - \frac{1}{3}(1)^3(0)^3\\ =& 0\end{aligned}$$

5. Appendix: Proof of FTC

Consider

$$\int_a^b \frac{d}{dt} f(r(t))dt$$

On the one hand, this equals

$$\int_a^b \frac{d}{dt} f(r(t))dt = f(r(b)) - f(r(a))$$

On the other hand, by the Chen Lu (Chain Rule):

$$\begin{aligned}
\frac{d}{dt} f(r(t)) =& \frac{d}{dt} f(x(t), y(t)) \\
=& \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} \\
=& \left(f_x\right)\left(x'(t)\right) + \left(f_y\right)\left(y'(t)\right) \\
=& \langle f_x, f_y \rangle \cdot \langle x'(t), y'(t) \rangle \\
=& \nabla f(x(t), y(t)) \cdot r'(t) \\
=& \nabla f(r(t)) \cdot r'(t)
\end{aligned}$$

Therefore:

$$\int_a^b \frac{d}{dt} f(r(t))dt = \int_a^b \nabla f(r(t)) \cdot r'(t) = \int_C \nabla f \cdot dr$$

Combining the two, we get:

$$\int_C \nabla f \cdot dr = \int_a^b \frac{d}{dt} f(r(t))dt = f(r(b)) - f(r(a))$$