LECTURE 13: FTC FOR LINE INTEGRALS (II)

Today: More practice with the FTC for line integrals and some interesting geometric insight

Recall: FTC for Line Integrals:

$$\int_C \nabla f \cdot dr = f(end) - f(start)$$

1. Examples

Video: FTC Example

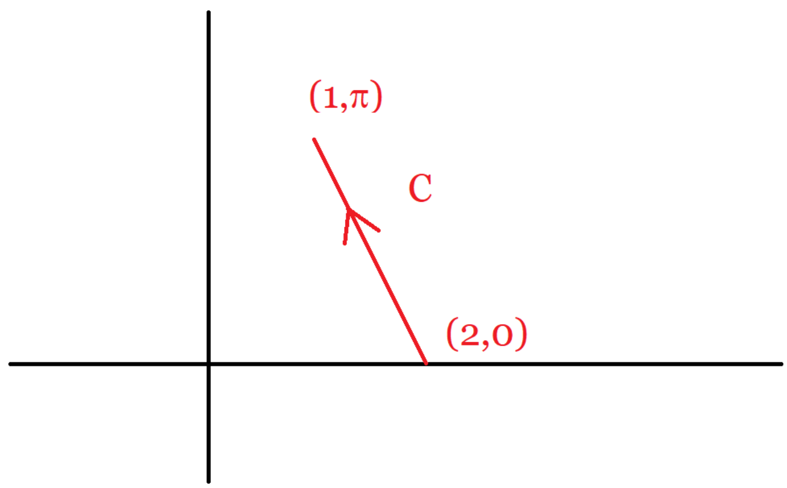
Example: $\int_C F \cdot dr$

$$F(x, y) = \underbrace{\langle \sin(y), x\cos(y) - \sin(y) \rangle}_{\langle P, Q \rangle}$$

C : line from $(2, 0)$ to $(1, \pi)$

(1) **Picture:**

Date: Monday, February 3, 2020.



(2) **Conservative:**

$$\begin{aligned} P_y =& (\sin(y))_y = \cos(y) \\ Q_x =& (x\cos(y) - \sin(y))_x = \cos(y) \\ P_y =& Q_x \checkmark \end{aligned}$$

(3) **Antiderivative:**

$$F = \nabla f \Rightarrow \langle \sin(y), x\cos(y) - \sin(y) \rangle = \langle f_x, f_y \rangle$$

$$f_x = \sin(y) \Rightarrow f(x,y) = \int \sin(y)dx = x\sin(y) + \text{ JUNK}$$

$$\begin{aligned} f_y =& x\cos(y) - \sin(y) \\ \Rightarrow& f(x,y) = \int x\cos(y) - \sin(y)dy = x\sin(y) + \cos(y) + JUNK \end{aligned}$$

$$f(x, y) = x\sin(y) + \cos(y)$$

(4)

$$\begin{aligned}
\int_C F \cdot dr = & \int_C \nabla f \cdot dr \\
= & f(1, \pi) - f(2, 0) \\
= & 1\sin(\pi) + \cos(\pi) - 2\sin(0) - \cos(0) \\
= & -1 - 1 \\
= & -2
\end{aligned}$$

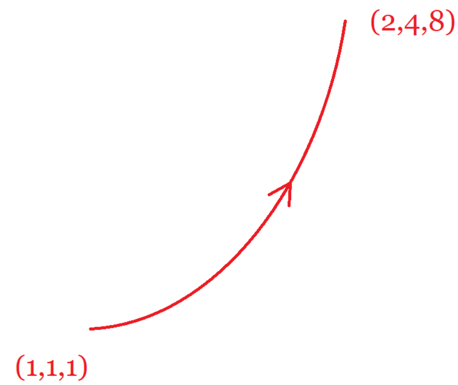
Video: FTC 3 Dimensions
Example: $\int_C F \cdot dr$

$F(x, y, z) = \langle yze^{xz}, e^{xz}, xye^{xz} \rangle$

C :

$$\begin{cases}
x(t) = t \\
y(t) = t^2 \\
z(t) = t^3 \\
1 \leq t \leq 2
\end{cases}$$

(1) **Picture:**



(2) **Conservative:** See Section 16.5

(3) **Antiderivative:**

$$F = \nabla f \Rightarrow \langle yze^{xz}, e^{xz}, xye^{xz} \rangle = \langle f_x, f_y, f_z \rangle$$

$$f_x = yze^{xz} \Rightarrow f(x,y,z) = \int yze^{xz} dx = yz\frac{e^{xz}}{z} + \text{ JUNK } = ye^{xz} + \text{ JUNK}$$

$$f_y = e^{xz} \Rightarrow f(x,y,z) = \int e^{xz} dy = ye^{xz} + \text{ JUNK}$$

$$f_z = xye^{xz} \Rightarrow f(x,y,z) = \int xye^{xz} dz = xy\frac{e^{xz}}{x} + \text{ JUNK } = ye^{xz} + \text{ JUNK}$$

$$f(x,y,z) = ye^{xz}$$

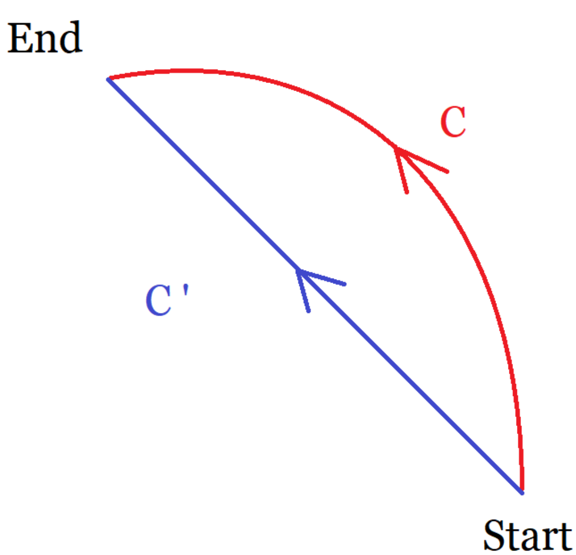
(4)

$$
\begin{aligned}
\int_C F \cdot dr = & \int_C \nabla f \cdot dr \\
= & f(2,4,8) - f(1,1,1) \\
= & 4e^{(2)(8)} - 1e^{(1)(1)} \\
= & 4e^{16} - e
\end{aligned}
$$

2. Path (in)-dependence

Video: Path Independence

Recall: In general, $\int_C F \cdot dr$ depends on the path C

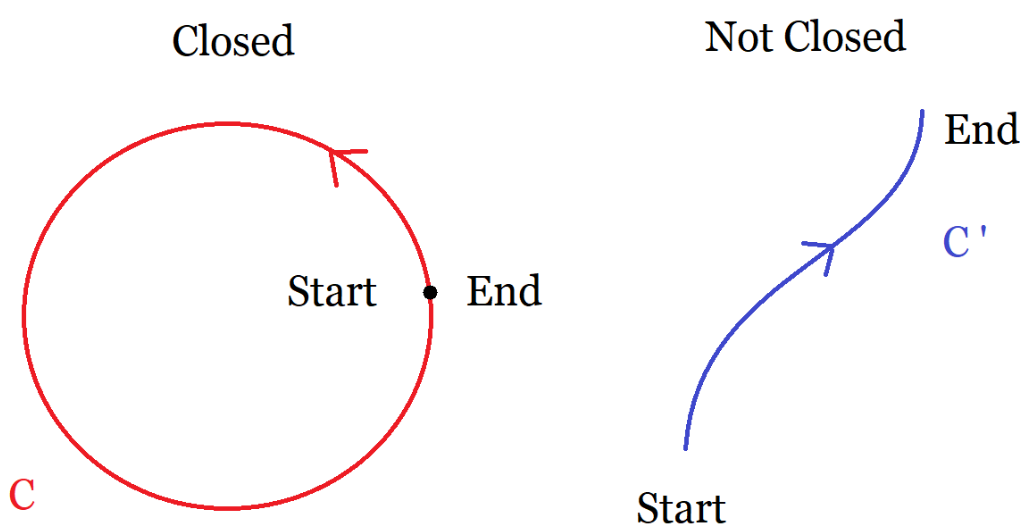


$\int_C F \cdot dr \neq \int_{C'} F \cdot dr$, even if C and C' have the same start/endpoints.

Question: When is $\int_C F \cdot dr$ **in**dependent of the path?

To figure this out, we'll need a quick definition:

Definition: C is **closed** if Start = End

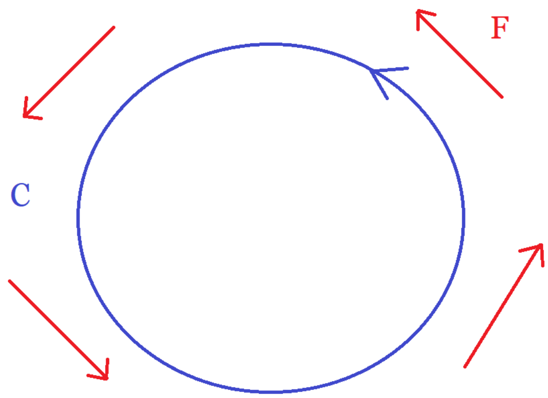


The following is an easy test for path independence:

Neat Fact:

$$\int_C F \cdot dr \text{ is independent of path } \Leftrightarrow \int_C F \cdot dr = 0 \text{ for every } \underline{\text{closed}}\ C$$

Nonexample:



$\int_C F \cdot dr \neq 0$, so not independent of path!

Note: The proof of the neat fact is interesting: For $\Rightarrow$, you consider the constant path, and for $\Leftarrow$, you loop around. See the end of the notes for a detailed proof

There's an even easier and more important test:

Important Fact:

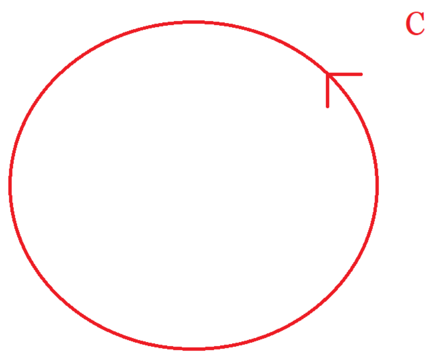
$$\int_C F \cdot dr \text{ is independent of path } \Leftrightarrow F \text{ is conservative } (F = \nabla f)$$

Which explains YET AGAIN why conservative vector fields are important!

Why?

($\Rightarrow$) Skip (but it explicitly constructs f)

($\Leftarrow$) Suppose $F = \nabla f$ and assume C is closed

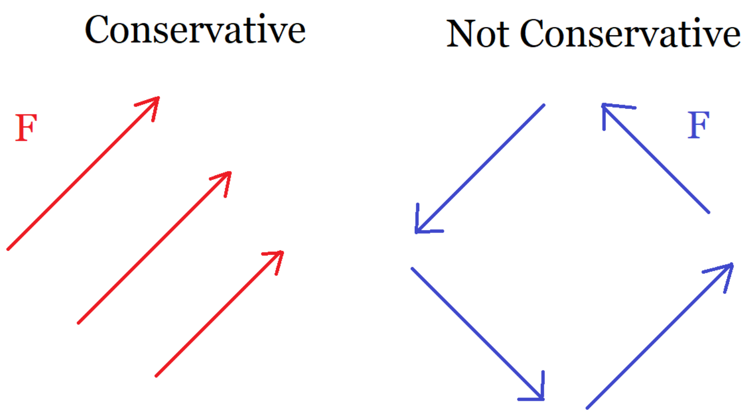


Then:

$$\int_C F \cdot dr = \int_C \nabla f \cdot dr = f(end) - f(start) = 0 \text{ (since } C \text{ is closed)}$$

Therefore $\int_C F \cdot dr = 0$ is closed for all C, and hence we're done by the Neat Fact.

Remark: For closed C, $\int_C F \cdot dr$ is sometimes called the circulation of F around C and measures how many times F loops around C. For conservative F, $\int_C F \cdot dr = 0$, so conservative F are irrotational.



Summary: Conservative vector fields are nice because:

(1) $F = \nabla f$ (they have antiderivatives)

(2) $\int_C F \cdot dr$ is easy to calculate (by FTC)

(3) $\int_C F \cdot dr$ is independent of path

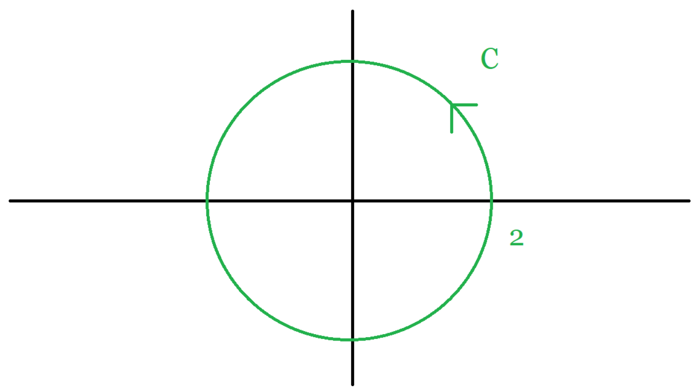
(4) F is irrotational

3. Pitfalls

Video: FTC Pitfalls

Example: $\int_C F \cdot dr$, $F(x,y) = \langle 2y, 3x \rangle$, C : Circle centered at $(0,0)$ of radius 2 (counterclockwise)

(1) **Picture:**



(2) **Conservative:**

$$\begin{aligned} P_y &= 2 \\ Q_x &= 3 \\ P_y &\neq Q_x \\ &\text{NO} \end{aligned}$$

RUH-OH!!! Well, in that case you have to get your hands dirty and calculate the integral.

(3) **Parametrize:**

$$\begin{cases} x(t) = & 2\cos(t) \\ y(t) = & 2\sin(t) \\ 0 \leq t \leq 2\pi \end{cases}$$

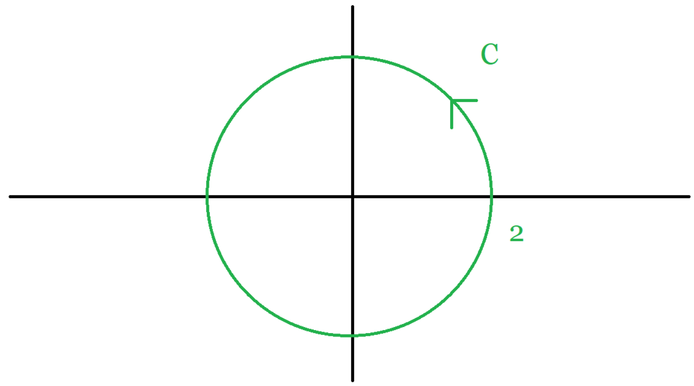
(4) **Integrate:**

$$\begin{aligned} \int_C F \cdot dr &= \int_0^{2\pi} F(r(t)) \cdot r'(t) \\ &= \int_0^{2\pi} \langle 2(2\sin(t)), 3(2\cos(t)) \rangle \cdot \langle -2\sin(t), 2\cos(t) \rangle \\ &= \int_0^{2\pi} -8\sin^2(t) + 12\cos^2(t) dt \\ &= \cdots \\ &= 4\pi \end{aligned}$$

Notice: $\int_C F \cdot dr \neq 0$ (even though C is closed), yet another argument why F is not conservative.

Example: Same, but $F(x, y) = \left\langle 2xy, x^2 \right\rangle$

(1) **Picture:**



(2) **Conservative:**

$$\begin{aligned} P_y &= 2x \\ Q_x &= 2x \\ P_y &= Q_x \ \checkmark \end{aligned}$$

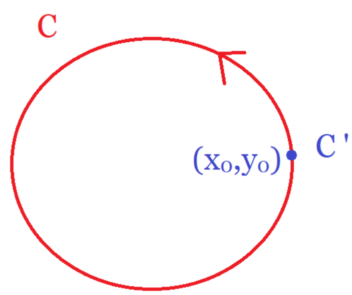
But since C is closed, we automatically get $\int_C F \cdot dr = 0$.

4. Optional Appendix: Proof of Neat Fact

Neat Fact:

$$\int_C F \cdot dr \text{ is independent of path } \Leftrightarrow \int_C F \cdot dr = 0 \text{ for every } \underline{\text{closed}}\ C$$

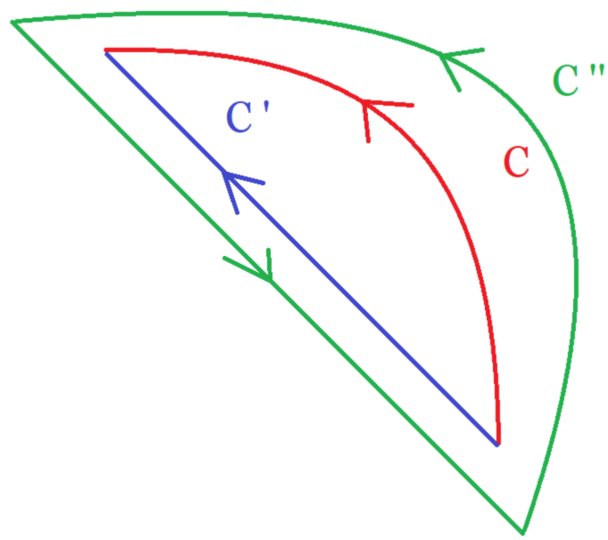
($\Rightarrow$) Suppose the integral is independent of path and let C be any closed curve. Pick any point (x_0, y_0) on C and let C' be the point (x_0, y_0) parametrized by $r(t) = \langle x_0, y_0 \rangle$.



Then $\int_{C'} F \cdot dr = 0$ (since $r(t) = \langle x_0, y_0 \rangle$ and $r'(t) = \langle 0, 0 \rangle$). Since the integral is independent of path, we get

$$\int_C F \cdot dr = \int_{C'} F \cdot dr = 0$$

($\Leftarrow$) Suppose C and C' are two curves with the same start and end-points, we want to show $\int_C F \cdot dr = \int_{C'} F \cdot dr$ and let C'' be C followed by C' (but in the opposite direction). So C'' is the loop formed by C and C'



Then, since C'' is closed, by assumption we get:

$$\begin{aligned}
\int_{C''} F \cdot dr &= 0 \\
\int_C F \cdot dr + \int_{-C'} F \cdot dr &= 0 \quad -C' \text{ is C' but in the other direction} \\
\int_C F \cdot dr - \int_{C'} F \cdot dr &= 0 \\
\int_C F \cdot dr &= \int_{C'} F \cdot dr
\end{aligned}$$