LECTURE 15: GREEN'S THEOREM (I)

Congrats on surviving the first boss, and welcome to the second part of Super Calculus 2E. They say the grass is greener on the other side, but I say that the grass is Green's Theorem-er on the other side, because today is all about Green's Theorem!

1. Motivation

Recall:

$$\text{FTC: } \int_C \nabla f \cdot dr = f(end) - f(start)$$

Problem: The FTC only works for conservative vector fields ($F = \nabla f$) Is there an FTC for non-conservative F? YES

WARNING: Today only works for 2 dimensions (we'll see 3D analogs in the coming weeks)

Motivation:

$$\text{2B: } \int F(x)dx = \int \int F'(x)dx$$

The 2E analog of this is

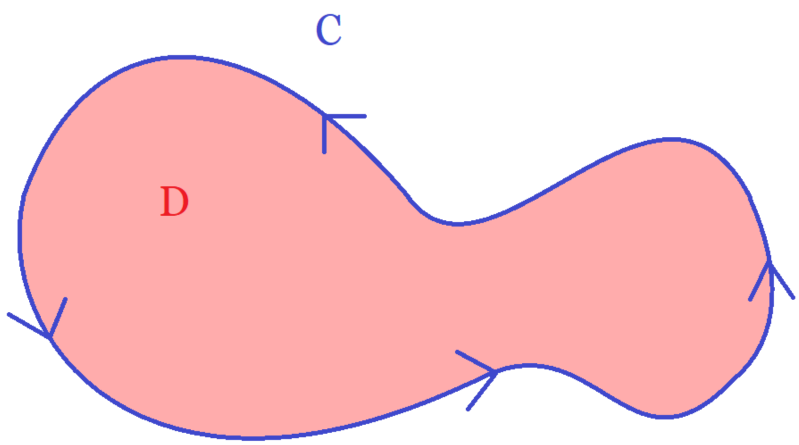
$$\text{2E: } \int_C F \cdot dr = \cancel{\int \int \nabla F dx}$$

Date: Monday, February 10, 2020.

The right-hand-side makes no sense! Need a scalar that says "The derivative of F" It turns out that the answer is Quixotic Peyams:

GREEN'S THEOREM: If C is a closed curve and D the region inside it, and $F = \langle P, Q \rangle$, then

$$\int_C F \cdot dr = \int \int_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dx dy$$



The idea is that instead of calculating a hard line integral (left), you calculate an easier double integral.

Mnemonic: QuiXotic PeYams

$$\text{OR: } \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{vmatrix} = \frac{\partial}{\partial x} Q - \frac{\partial}{\partial y} P = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

Note: Green's Theorem because George Green, not because of the color green

2. Examples

Video: Green's Theorem

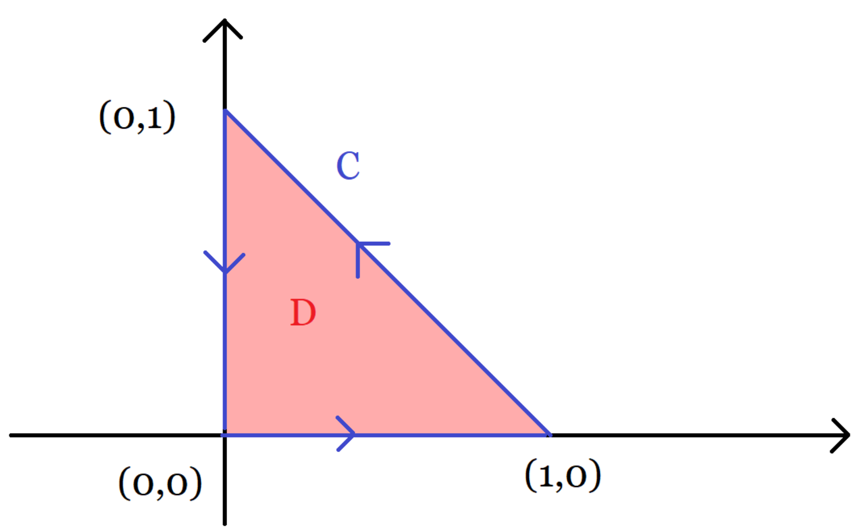
Again, useful for calculating line integrals.

Example 1: $\int_C F \cdot dr$

$$F(x,y) = \langle x^4, xy \rangle$$

C : Triangle connecting $(0,0), (1,0), (0,1)$

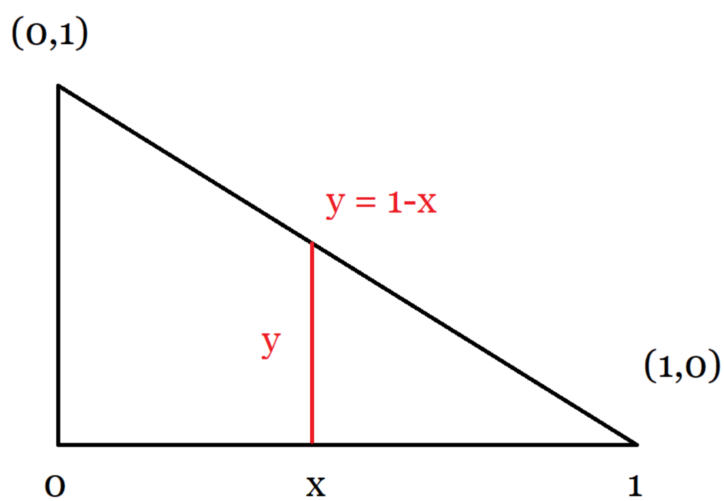
(1) **Picture:**



Notice: It's a **PAIN** to do the line integral directly! Not only is F complicated, but you also need to split the line integral up into 3 pieces, ughhh!!!

(2)

$$\begin{aligned}\int_C F \cdot dr &= \int\int_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dxdy \\ &= \int\int_D (xy)_x - \left(x^4\right)_y dxdy \\ &= \int\int_D ydxdy\end{aligned}$$



$$\begin{aligned}0 \leq y &\leq 1 - x \\ 0 \leq x &\leq 1\end{aligned}$$

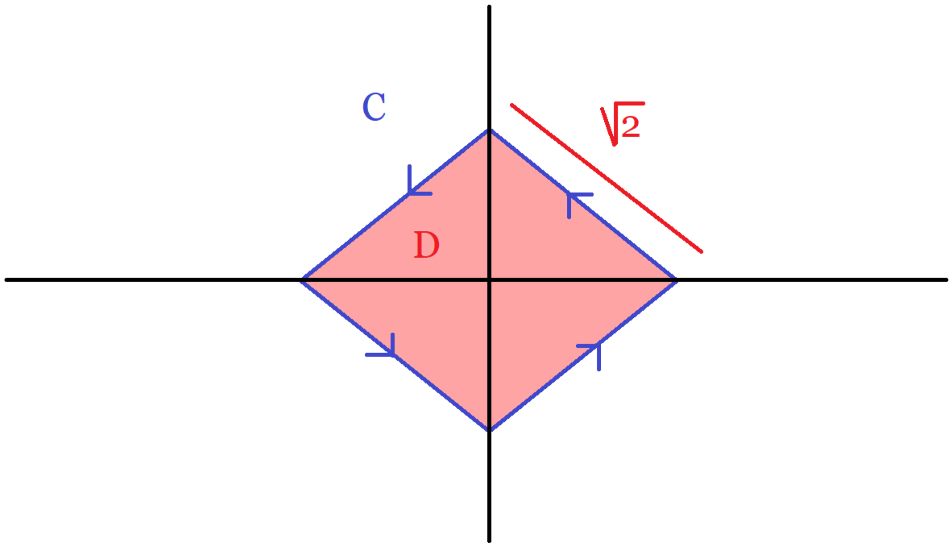
$$
\begin{aligned}
&= \int_0^1 \int_0^{1-x} y dy dx \\
&= \int_0^1 \left[\frac{y^2}{2} \right]_{y=0}^{y=1-x} \\
&= \int_0^1 \frac{(1-x)^2}{2} dx \\
&= \cdots \\
&= \frac{1}{6}
\end{aligned}
$$

Example 2: $\int_C F \cdot dr$

$$F(x,y) = \left\langle 3y - e^{\sin(x)}, 7x + \sqrt{y^4 + 1} \right\rangle$$

C : Square with vertices $(\pm 1, 0), (0, \pm 1)$

(1) **Picture:**



(2)

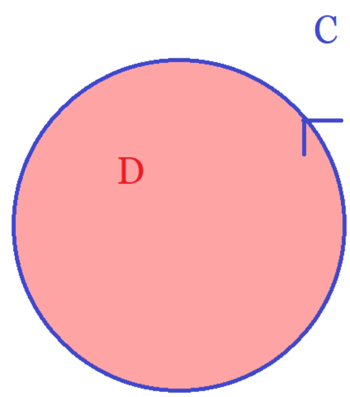
$$\begin{aligned}\int_C F \cdot dr &= \int\int_D \frac{7x+\sqrt{y^4+1}}{\partial x} - \frac{3y-e^{\sin(x)}}{\partial y} dxdy \\ &= \int\int_D 7-3dxdy \\ &= \int\int_D 4dxdy \\ &=4\int\int_D 1dxdy \\ &=4 \text{ Area } (D) \\ &=4(\sqrt{2})^2 \\ &=8\end{aligned}$$

Example 3: What is F is conservative?

$$F(x,y) = \langle xy^2, x^2y \rangle$$

C : Circle of Radius 1

(1) **Picture:**



(2)

$$
\begin{aligned}
\int_C F \cdot dr &= \int\int_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dxdy \\
&= \int\int_D \left(x^2 y\right)_x - \left(xy^2\right)_y dxdy \\
&= \int\int_D 2xy - 2xydxdy \\
&= \int\int_D 0dxdy \\
&=0
\end{aligned}
$$

(So if C is closed and F is conservative, don't even need to use FTC, just use Green's Theorem!)

Remark:

Showed that $F = \langle P, Q \rangle$ conservative $\Rightarrow P_y = Q_x$ (using Clairaut's Theorem)

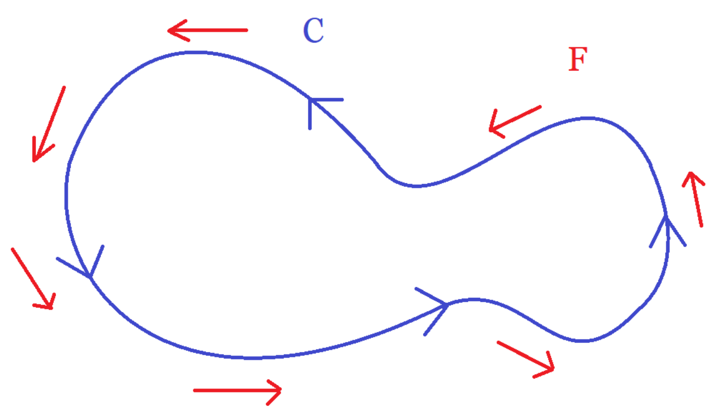
Now if $P_y = Q_x$ (and no holes), then for all closed C,

$$\int_C F \cdot dr = \int \int_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$$

So F is conservative (by neat fact from last time)

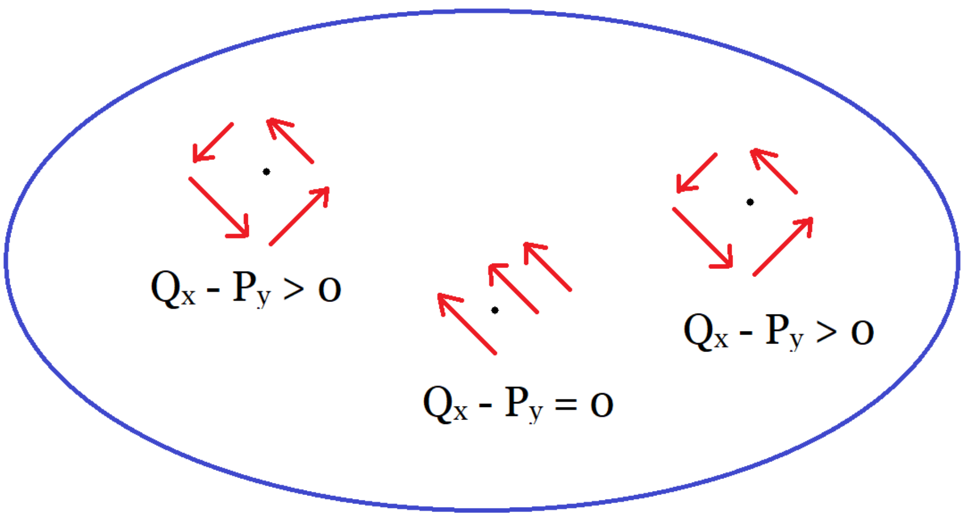
So F conservative $\Leftrightarrow P_y = Q_x$

3. Some Intuition



$\int_C F \cdot dr$ measures the circulation of F around C (think F = wind or water) which you can think of a **macroscopic rotation**

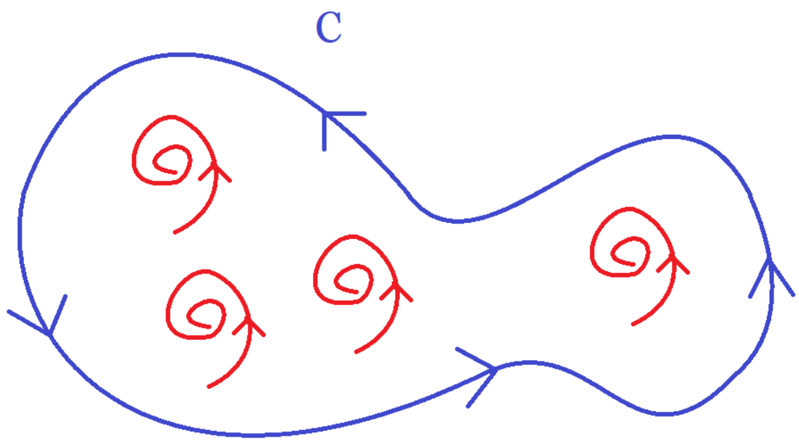
$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ measures the rotation of F around a point, which is a **microscopic rotation**



Green's Theorem says:

$$\underbrace{\int\int \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dxdy}_{\text{Sum of Microscopic Rotations}} = \underbrace{\int_C F \cdot dr}_{\text{Macroscopic Rotation}}$$

Which kind of makes sense! Think of the microscopic rotations as mini-whirlpools (or hurricanes) in a bath-tub C



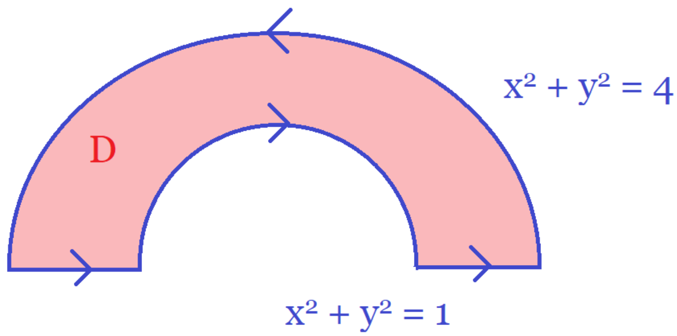
Green's Theorem says that if you add up all the whirlpools inside the bathtub, you get a gigantic whirlpool/circulation around C

4. One More Example (if time permits)

Example 4: $\int_C y^2 dx + 3xy dy$

Means: $\int_C F \cdot dr$, $F = \langle y^2, 3xy \rangle$

C : Boundary of the region $1 \leq x^2 + y^2 \leq 4$ in the upper-half-plane

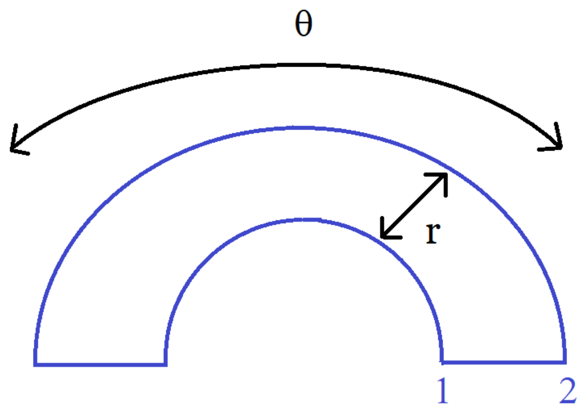


(1)

(Strictly speaking, notice that C goes clockwise at some point, but don't worry about it)

(2)

$$
\begin{aligned}
&\int_C y^2 dx + 3xy dy \\
=&\int\int_D \frac{\partial\,(3xy)}{\partial x} - \frac{\partial\left(y^2\right)}{\partial y} dxdy \\
=&\int\int_D 3y - 2y dxdy \\
=&\int\int_D y dxdy
\end{aligned}
$$



$$1 \leq r \leq 2$$
$$0 \leq \theta \leq \pi$$

$$\begin{aligned}
&= \int_0^{\pi} \int_1^2 r \sin(\theta) r dr d\theta \\
&= \left(\int_1^2 r^2 dr \right) \left(\int_0^{\pi} \sin(\theta) d\theta \right) \\
&= \left(\frac{7}{3} \right) (2) \\
&= \frac{14}{3}
\end{aligned}$$