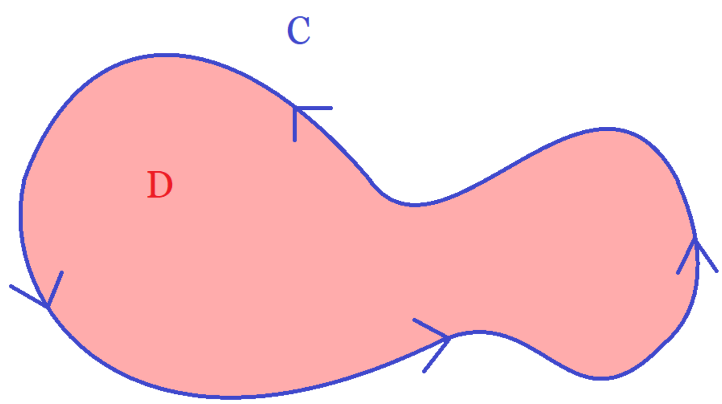
LECTURE 16: GREEN'S THEOREM (II)

Welcome to the second part of our Green's Theorem extravaganza! Today is all about applications of Green's Theorem.



Green's Theorem

$$\int_C F \cdot dr = \int \int_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dxdy$$

Last time, we saw that Green's Theorem helps us simplify line integrals. Now you may ask: Is the opposite true? Could we use Green's theorem to simplify double integrals? Not really **except** for one special case:

Date: Wednesday, February 12, 2020.

1. Area 51

Recall

$$\text{Area } (D) = \int\int_D 1 \, dxdy$$

So **IF** $F = \langle P, Q \rangle$ is such that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$, then:

$$\begin{aligned}\int_C F \cdot dr &\overset{G}{=} \int\int_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dxdy \\ &= \int\int_D 1 dxdy \\ &= \text{Area } (D)\end{aligned}$$

Many choices for P and Q such that $Q_x - P_y = 1$

(Examples: $P = 0, Q = x$ or $P = -y, Q = 0$)

"Best" choice: $P = -\frac{y}{2}, Q = \frac{x}{2}$, which gives:

$$F = \langle P, Q \rangle = \left\langle -\frac{y}{2}, \frac{x}{2} \right\rangle = \frac{1}{2} \langle -y, x \rangle \to \frac{1}{2} (-ydx + xdy)$$

FACT (Memorize)

$$\text{Area } (D) = \frac{1}{2} \int_C xdy - ydx$$

Mnemonic: $\frac{1}{2} \begin{vmatrix} x & y \\ dx & dy \end{vmatrix} = \frac{1}{2} (xdy - ydx)$

2. OMG Example

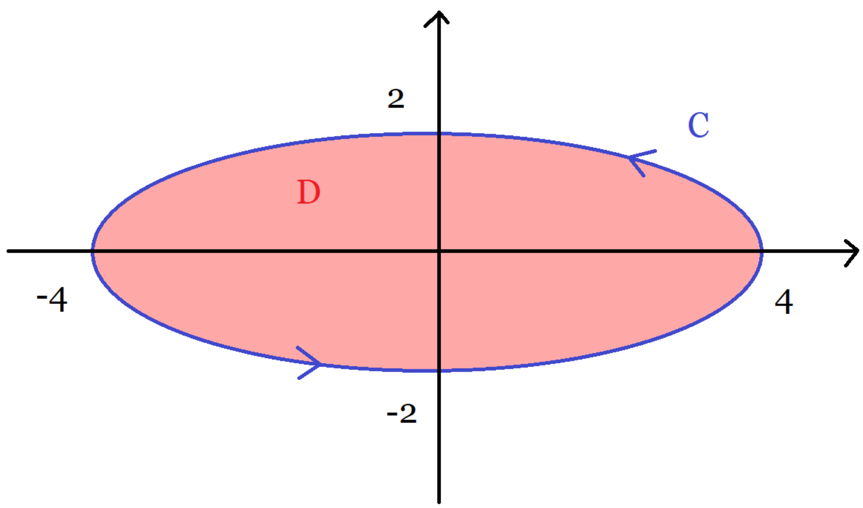
Video: Area of Ellipse

Example

Find the area enclosed by the ellipse

$$\frac{x^2}{4^2} + \frac{y^2}{2^2} = 1$$

(1) **Picture:**



(2)

$$\begin{cases} x(t) = 4\cos(t) \\ y(t) = 2\sin(t) \\ \quad 0 \leq t \leq 2\pi \end{cases}$$

(3)

$$\begin{aligned}
\text{Area } (D) =& \frac{1}{2} \int_C x \frac{dy}{dt} - y \frac{dx}{dt} dt \\
=& \frac{1}{2} \int_0^{2\pi} x(t)y'(t) - y(t)x'(t) dt \\
=& \frac{1}{2} \int_0^{2\pi} 4\cos(t) 2\cos(t) - 2\sin(t) \left(-4\sin(t)\right) dt \\
=& \frac{1}{2} \int_0^{2\pi} \underbrace{8\cos^2(t) + 8\sin^2(t)}_{8} dt \\
=& \left(\frac{1}{2}\right) (8) (2\pi) \\
=& 8\pi
\end{aligned}$$

OMG, look how effortless this was!

3. OMGGG Example

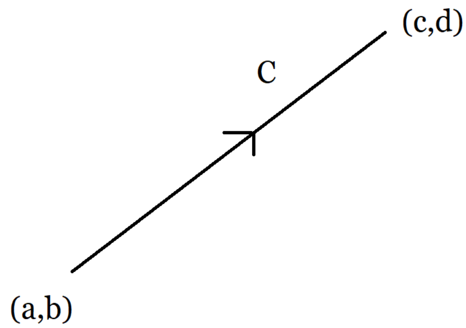
Video: Area of a Polygon

You might say "OMG Peyam, there's no way this could be even more exciting!!!" Oh, just wait for it! ☺

Example

(a) Prep: Find $\int_C x dy - y dx$, C : Line connecting (a, b) to (c, d)

(1) **Picture:**



(2) **Parametrize:**

$$\begin{cases} x(t) = (1-t)a + tc = & a + t(c-a) \\ y(t) = (1-t)b + td = & c + t(d-b) \\ \quad 0 \leq t \leq 1 \end{cases}$$

(3)

$$\begin{aligned} \int_C x dy - y dx &= \int_0^1 x(t)y'(t) - y(t)x'(t) dt \\ &= \int_0^1 (a + t(c-a))(d-b) - (b + t(d-b))(c-a) dt \\ &= \int_0^1 a(d-b) + \cancel{t(c-a)(d-b)} - b(c-a) - \cancel{t(d-b)(c-a)} dt \\ &= \int_0^1 ad - \cancel{ab} - bc + \cancel{ab} dt \\ &= ad - bc \\ &= \begin{vmatrix} a & b \\ c & d \end{vmatrix} \end{aligned}$$

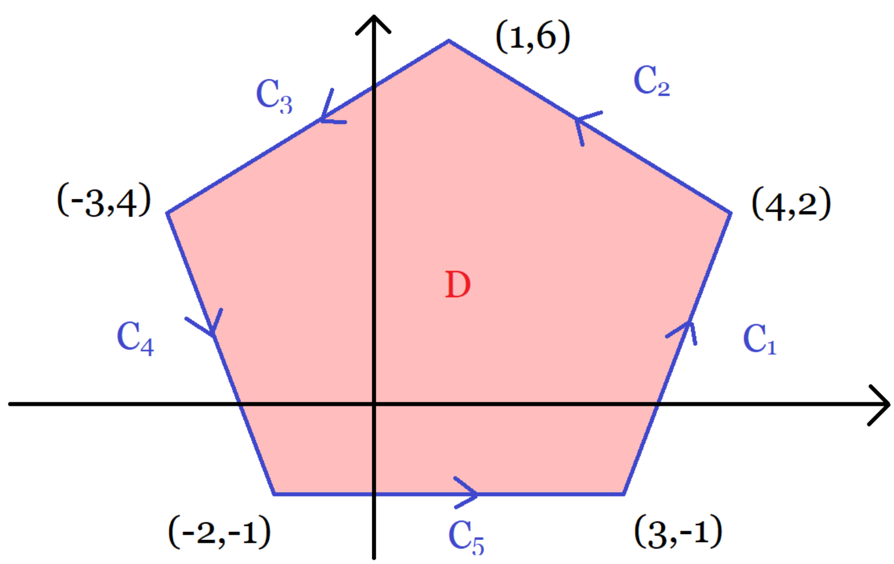
Therefore:

$$\int_C xdy - ydx = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

OMG Part

(b) Find the area of the pentagon with vertices $(3,-1),(4,2),(1,6),(-3,4),(-2,-1)$

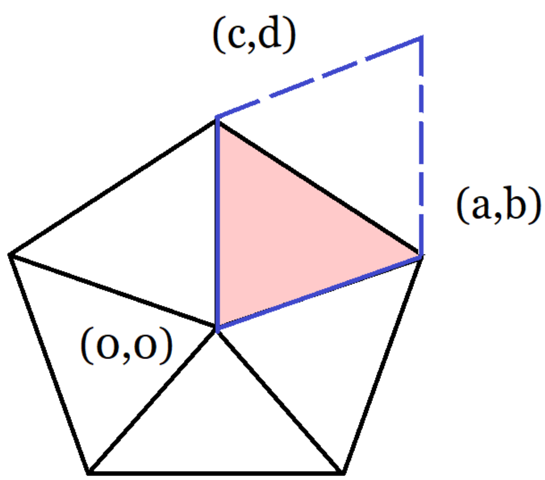
(In fact, any polygon works)

(1) 

(2)

$$
\begin{aligned}
\text{Area } (D) =& \frac{1}{2}\int_C xdy - ydx \\
=& \frac{1}{2}\left(\int_{C_1} xdy - ydx + \int_{C_2} xdy - ydx + \cdots + \int_{C_5} xdy - ydx\right) \\
=& \frac{1}{2}\left(\begin{vmatrix} 3 & -1 \\ 4 & 2 \end{vmatrix} + \begin{vmatrix} 4 & 2 \\ 1 & 6 \end{vmatrix} + \begin{vmatrix} 1 & 6 \\ -3 & 4 \end{vmatrix} + \begin{vmatrix} -3 & 4 \\ -2 & -1 \end{vmatrix} + \begin{vmatrix} -2 & -1 \\ 3 & -1 \end{vmatrix}\right) \\
=& \frac{1}{2}(10 + 22 + 22 + 11 + 5) \\
=& 35 \text{ BOOM!!!}
\end{aligned}
$$

Why this works?



A pentagon (or any polygon) is the sum of triangles, which are half-parallelograms, so

$$\begin{aligned}\text{Area(Pentagon)} &= \text{Sum of Areas of Triangles}\\ &= \frac{1}{2}\left(\text{ Sum of areas of Parallelograms }\right)\\ &= \frac{1}{2}\left(\text{Sum of }\begin{vmatrix} a & b \\ c & d\end{vmatrix}\right)\ \text{(from Math 3A)}\end{aligned}$$

Note: There is a 3 dimensional analog of this, you can check it out here: Volume of Polyhedron (but it requires a 3D version of Green's theorem from 16.9)

4. Mmmmmh, Donut Holes

Video: Winding Number

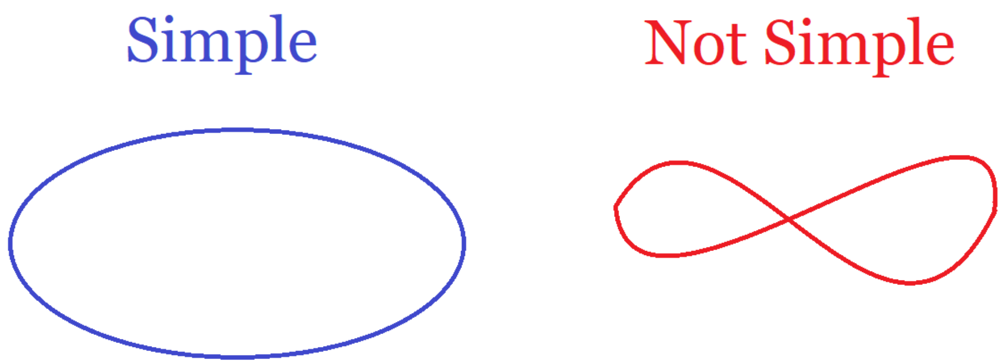
(This is not directly related to Green's Theorem, but you can use Green to prove this result)

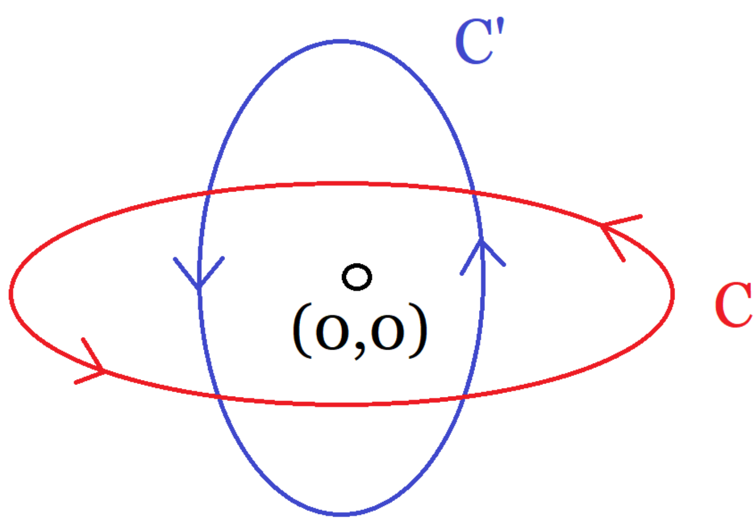
Suppose F is conservative, but undefined at $(0,0)$ (so there is a hole at $(0,0)$), like $F = \left\langle -\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$.

In that case, we don't have $\int_C F \cdot dr = 0$ any more, **but:**

Fact

$\int_C F \cdot dr$ is still independent of C, as long as C encloses $(0,0)$ (and C is simple = no crossings)





In the above picture $\int_C F \cdot dr = \int_{C'} F \cdot dr$

Example

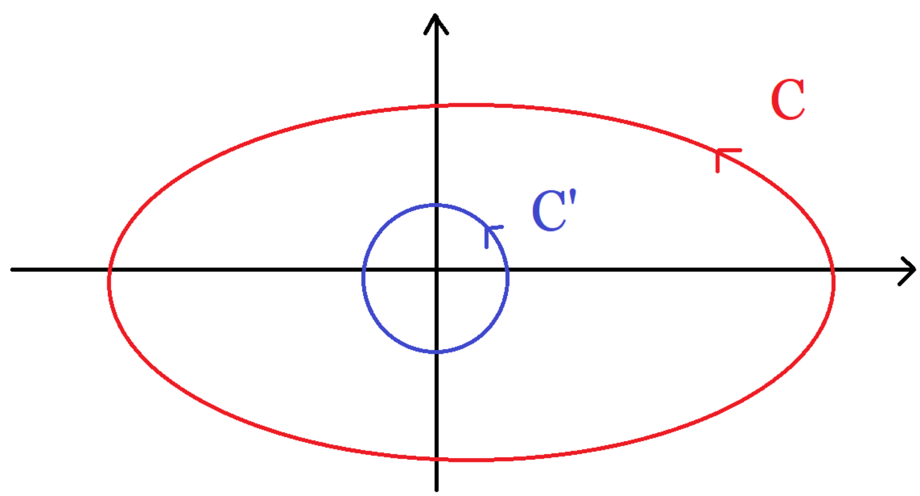
Calculate $\int_C F \cdot dr$

$$F = \left\langle -\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$$

And C is any simple curve enclosing $(0,0)$

(1) **Picture:**

Idea: Since the answer is the same anyway, choose the **easiest** curve enclosing $(0,0)$ (the one that simplifies F as much as possible)



So Let C' be the **circle** centered at $(0,0)$ and radius 1.

(2) **Parametrize** C':

$$\begin{cases} x(t) = \cos(t) \\ y(t) = \sin(t) \\ 0 \leq t \leq 2\pi \end{cases}$$

(3)

$$\begin{aligned} \int_{C'} F \cdot dr &= \int_0^{2\pi} \left\langle \frac{-\sin(t)}{\cos^2(t)+\sin^2(t)}, \frac{\cos(t)}{\cos^2(t)+\sin^2(t)} \right\rangle \cdot \langle -\sin(t), \cos(t) \rangle \, dt \\ &= \int_0^{2\pi} \sin^2(t) + \cos^2(t) dt \\ &= \int_0^{2\pi} 1 dt \\ &= 2\pi \end{aligned}$$

(4) By **Fact:**

$$\int_C F \cdot dr = \int_{C'} F \cdot dr = 2\pi$$

Remarks:

(1) The fact that we get a nonzero answer (even though F is conservative) should not be seen as a drawback, but as a feature. Already gives us information about the 'topology' of the domain (namely, here there's a hole)

(2) In fact, the whole field of complex analysis exists *because* the answer is nonzero! (wow)

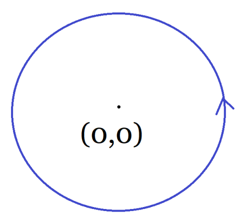
(3) For **any** closed curve C (not necessarily simple), $\frac{1}{2\pi} \int_C F \cdot dr$ with $F = \left\langle -\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$ is called the winding number of C

(the origin) and counts how many times C loops around $(0,0)$

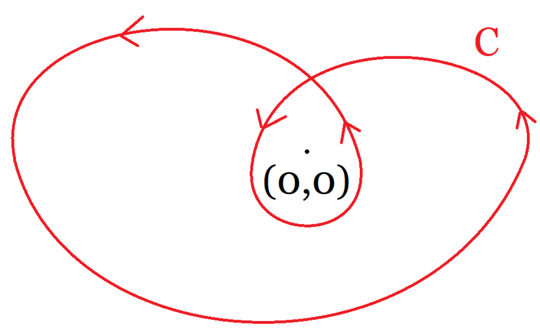
Example 1: For the circle C, the winding number is (by the above):

$$\frac{1}{2\pi}\int_C F\cdot dr = \frac{1}{2\pi}2\pi = 1$$

Which makes sense since the circle just loops around $(0,0)$ once



Example 2: In the following example, the winding number of C is 2, because C loops around $(0,0)$ twice.



(4) If you want to learn more about holes, you should check out the field of algebraic topology. In fact, you may have heard of the phrase "A donut is similar to a cup of coffee;" that comes from algebraic topology.