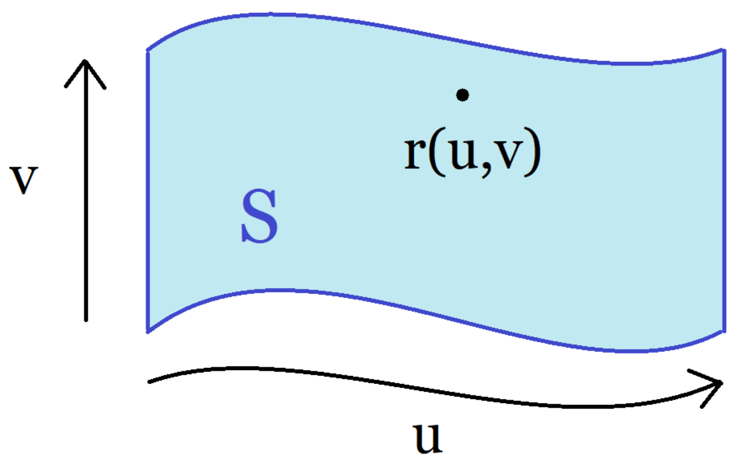
LECTURE 17: PARAMETRIC SURFACES (I)

1. Examples

The goal for the rest of the course is to generalize everything that we know about line integrals to surfaces.

Today's Goal: How to parametrize a surface S ?



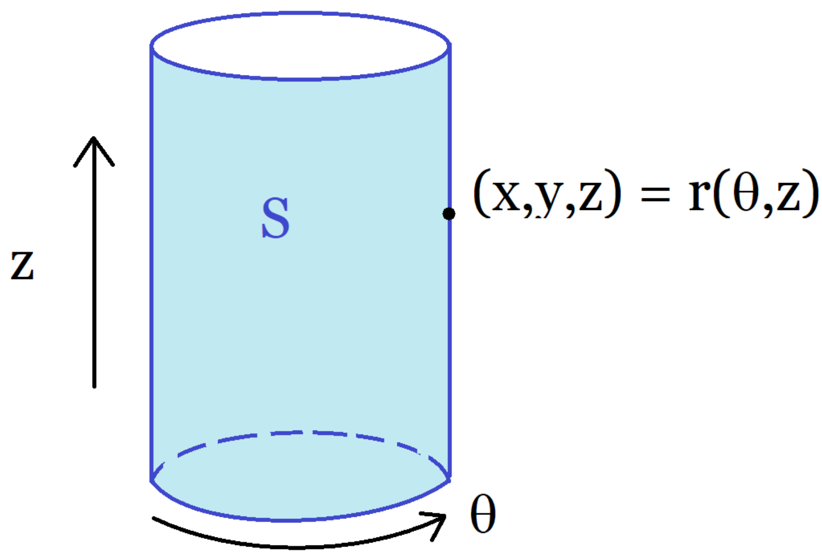
Since S is 2-dimensional, we need 2 parameters u and v. The analog of $r(t)$ is then $r(u, v)$.

Date: Friday, February 14, 2020.

Example 1

Parametrize the cylinder $x^2 + y^2 = 4$

It's just cylindrical coordinates with $r = 2$



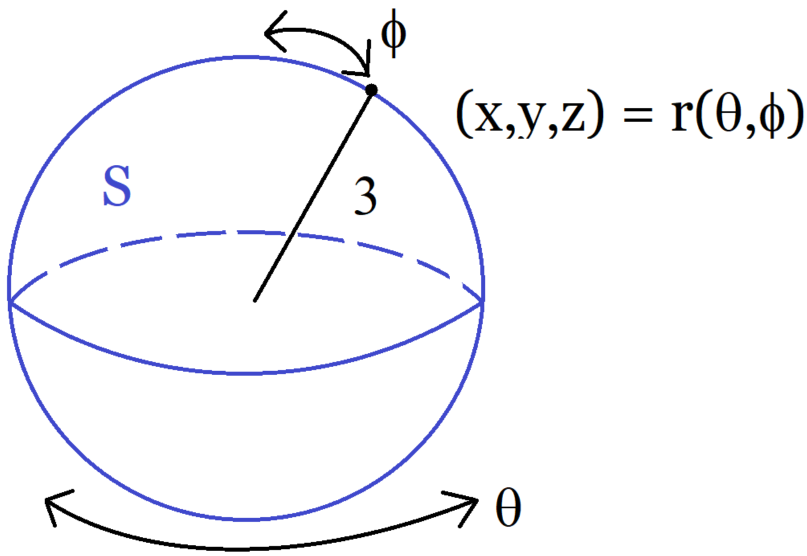
$$\begin{cases} x = & 2\cos(\theta) \\ y = & 2\sin(\theta) \\ z = & z \end{cases}$$

$$r(\theta, z) = \langle 2\cos(\theta), 2\sin(\theta), z\rangle$$

With $0 \leq \theta \leq 2\pi, -\infty < z < \infty$

Example 2

Parametrize the sphere $x^2 + y^2 + z^2 = 9$



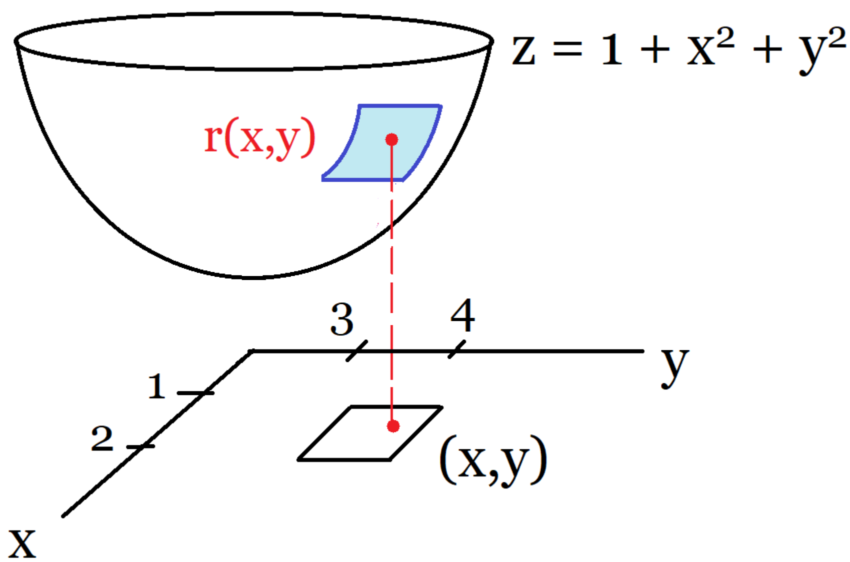
$$\begin{cases} x = 3\sin(\phi)\cos(\theta) \\ y = 3\sin(\phi)\sin(\theta) \\ z = 3\cos(\phi) \end{cases}$$

$$r(\theta, \phi) = \langle 3\sin(\phi)\cos(\theta), 3\sin(\phi)\sin(\theta), 3\cos(\phi)\rangle$$

With $0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi$

Example 3: Functions

Parametrize the portion of the paraboloid $z = 1 + x^2 + y^2$ over the rectangle $[1, 2] \times [3, 4]$



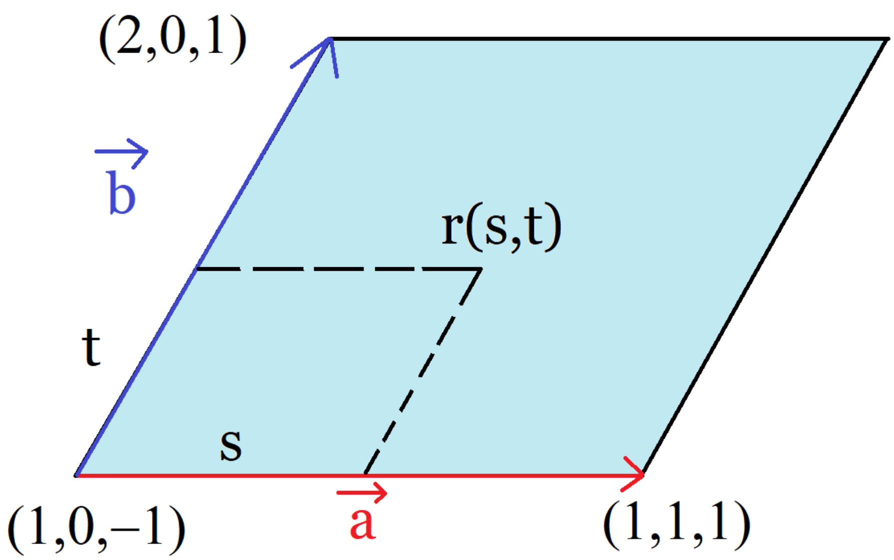
$$\begin{cases} x = x \\ y = y \\ z = 1 + x^2 + y^2 \end{cases} \implies \begin{array}{c} r(x,y) = \left\langle x, y, 1 + x^2 + y^2 \right\rangle \\ 1 \leq x \leq 2 \\ 3 \leq y \leq 4 \end{array}$$

More generally, for functions f, $r(x, y) = \langle x, y, f(x, y)\rangle$

(So surfaces are more general than functions)

Example 4: Planes (will probably skip)

Parametrize the plane containing $A = (1,0,-1), B = (1,1,1), C = (2,0,1)$



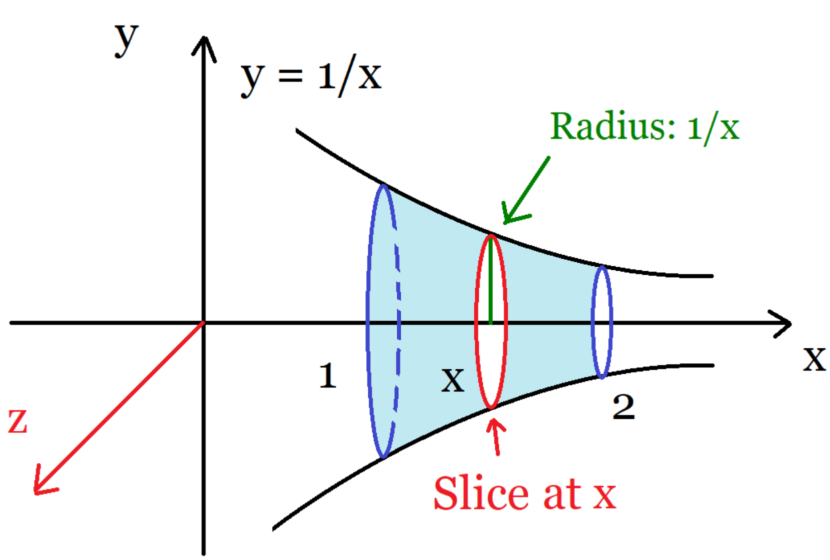
$$\mathbf{a} = \langle 1-1, 1-0, 1-(-1)\rangle = \langle 0,1,2\rangle$$
$$\mathbf{b} = \langle 2-1, 0-0, 1-(-1)\rangle = \langle 1,0,2\rangle$$

$$\begin{aligned} r(s,t) &= \underbrace{\langle 1,0,-1\rangle}_{\text{Start}} + s\mathbf{a} + t\mathbf{b} \\ &= \langle 1,0,-1\rangle + s\langle 0,1,2\rangle + t\langle 1,0,2\rangle \\ &= \langle 1+t, s, -1+2s+2t\rangle \\ &\quad -\infty < s < \infty, -\infty < t < \infty \end{aligned}$$

(If you've taken 3A, then S is a linear combo of $\mathbf{a}$ and $\mathbf{b}$)

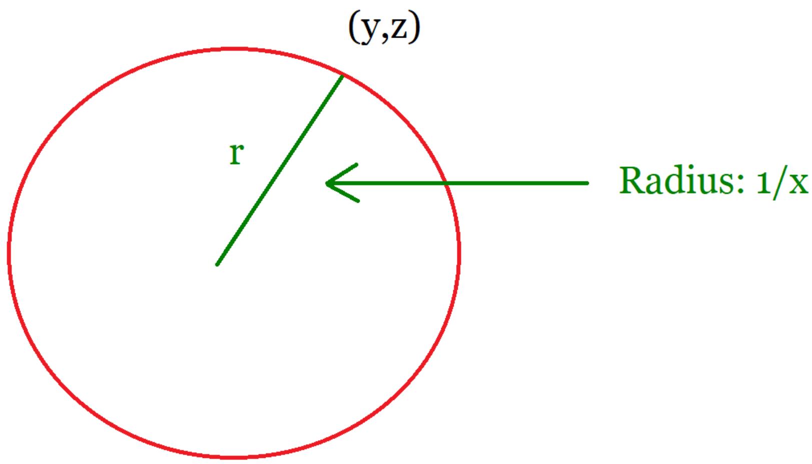
Example 5: Solids of Revolution (will probably skip)

(Math 2B) Parametrize the Surface obtained by rotating the curve $y = \frac{1}{x}$ between $x = 1$ and $x = 2$ about the $x-$axis



Start with $x = x$, $1 \leq x \leq 2$.

Look at slice at x:



$$\begin{cases} y = & r\cos(\theta) = \left(\frac{1}{x}\right)\cos(\theta) \\ z = & r\sin(\theta) = \left(\frac{1}{x}\right)\sin(\theta) \end{cases}$$

$$r(x,\theta) = \left\langle x, \left(\frac{1}{x}\right)\cos(\theta), \left(\frac{1}{x}\right)\sin(\theta) \right\rangle$$

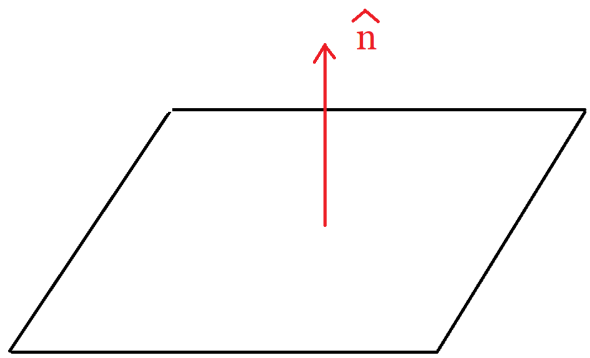
$$1 \leq x \leq 2$$

$$0 \leq \theta \leq 2\pi$$

More generally: If you rotate the function $f(x)$ about the $x-$axis, then $r(x,\theta) = \langle x, f(x)\cos(\theta), f(x)\sin(\theta)\rangle$

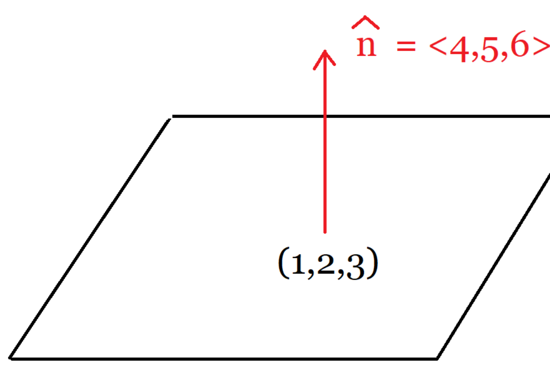
2. Quick Facts about Planes

Math 2D: The single, MOST IMPORTANT thing about a plane is the NORMAL VECTOR $\hat{n}$!!!



Example

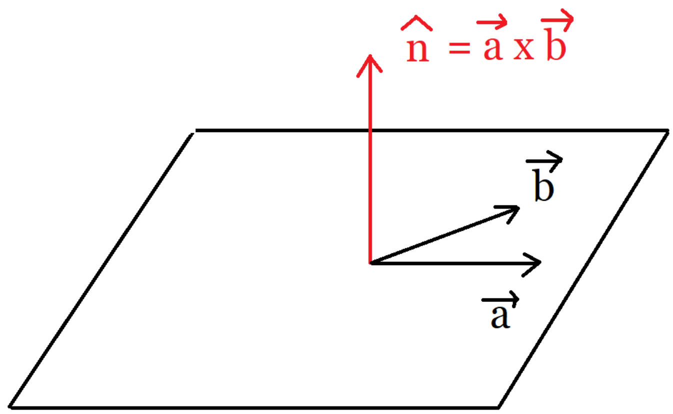
Find the equation of the plane going through $(1, 2, 3)$ and with normal vector $\hat{n} = \langle 4, 5, 6\rangle$



$$4(x-1)+5(x-2)+6(z-3)=0$$

THIS is why $\hat{n}$ is important; it allows us to easily find the equation of a plane.

Math 2D: If a plane contains vectors $\mathbf{a}$ and $\mathbf{b}$, then $\hat{n} = \mathbf{a} \times \mathbf{b}$

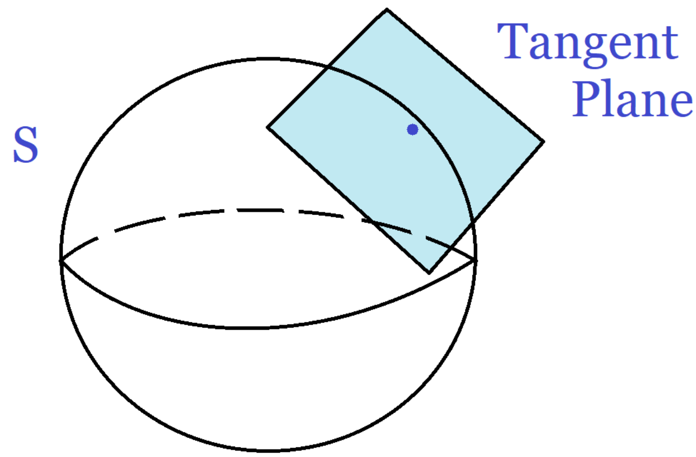


This is why the cross product is important; it allows us to find $\hat{n}$.

3. Tangent Planes

Video: Tangent Plane to a Surface

Goal: Find the tangent plane to a surface (just like we you did for functions in Math 2D)



Here is where parametrizations and normal vectors help us!

Example:

Find the equation of the tangent plane to

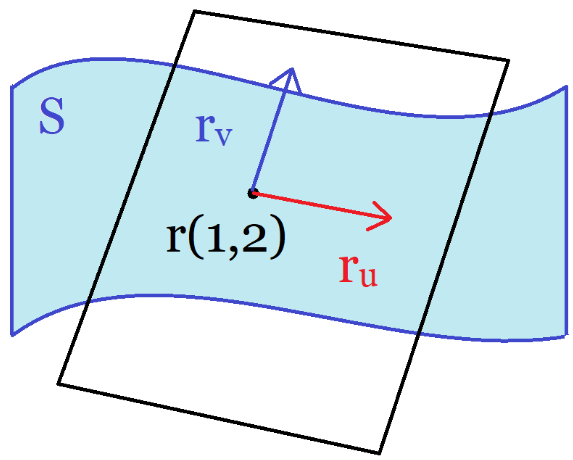
$$r(u,v) = \left\langle u^2+1, v^2+1, u+v \right\rangle$$

At $u=1, v=2$

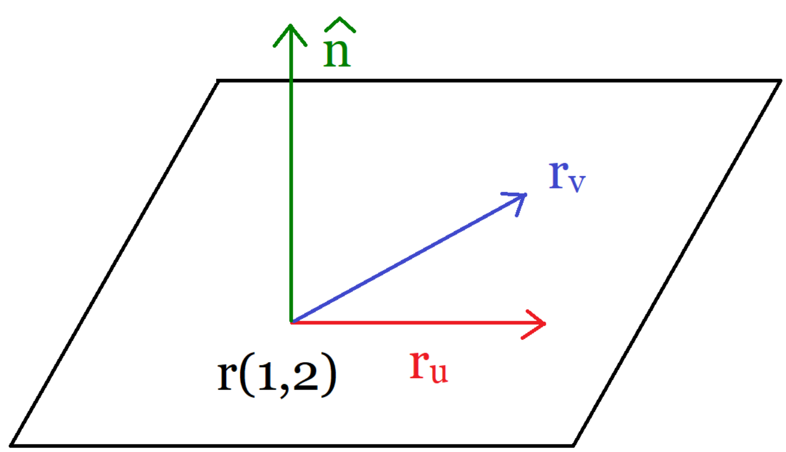
(1) **Slopes:** Calculate r_u and r_v (at $u=1, v=2$)

$$r_u = \left\langle (u^2+1)_u, (v^2+1)_u, (u+v)_u \right\rangle = \langle 2u, 0, 1\rangle = \langle 2, 0, 1\rangle \text{ (at } u=1, v=2)$$
$$r_v = \left\langle (u^2+1)_v, (v^2+1)_v, (u+v)_v \right\rangle = \langle 0, 2v, 1\rangle = \langle 0, 4, 1\rangle$$

Fact: r_u and r_v are on the tangent plane



(2) **Normal Vector:**



$$
\begin{aligned}
\hat{n} =& r_u \times r_v \\
=& \langle 2, 0, 1 \rangle \times \langle 0, 4, 1 \rangle \\
=& \begin{vmatrix} i & j & k \\ 2 & 0 & 1 \\ 0 & 4 & 1 \end{vmatrix} \\
=& \begin{vmatrix} 0 & 1 \\ 4 & 1 \end{vmatrix} i - \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} j + \begin{vmatrix} 2 & 0 \\ 0 & 4 \end{vmatrix} k \\
=& -4i - 2j + 8k \\
=& \langle -4, -2, 8 \rangle
\end{aligned}
$$

(3) **Point:**

$$r(1,2) = \left\langle 1^2 + 1, 2^2 + 1, 1 + 2 \right\rangle = \langle 2, 5, 3 \rangle$$

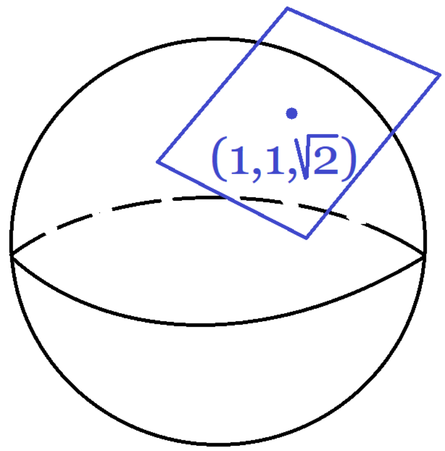
(4) **Equation:** $\hat{n} = \langle -4, -2, 8 \rangle$, Point $(2, 5, 3)$

$$-4(x-2) - 2(y-5) + 8(z-3) = 0$$

Example:

Same but for $x^2 + y^2 + z^2 = 4$ at $(1, 1, \sqrt{2})$

(1) **Picture:**



(2)

$$\begin{cases} x = & 2\sin(\phi)\cos(\theta) \\ y = & 2\sin(\phi)\sin(\theta) \\ z = & 2\cos(\phi) \end{cases}$$

$$r(\theta,\phi) = \langle 2\sin(\phi)\cos(\theta), 2\sin(\phi)\sin(\theta), 2\cos(\phi)\rangle$$

(3) Find θ and ϕ (analog of $u = 1$ and $v = 2$)

$$(2\sin(\phi)\cos(\theta), 2\sin(\phi)\sin(\theta), 2\cos(\phi)) = (1, 1, \sqrt{2})$$

$$\begin{cases} 2\cos(\phi) = \sqrt{2} \Rightarrow \cos(\phi) = \dfrac{1}{\sqrt{2}} \Rightarrow \phi = \dfrac{\pi}{4} \\ 2\sin(\phi)\cos(\theta) = 1 \Rightarrow 2\left(\dfrac{1}{\sqrt{2}}\right)\cos(\theta) = 1 \Rightarrow \quad \cos(\theta) = \dfrac{1}{\sqrt{2}} \\ 2\sin(\phi)\sin(\theta) = 1 \Rightarrow 2\left(\dfrac{1}{\sqrt{2}}\right)\sin(\theta) = 1 \Rightarrow \quad \sin(\theta) = \dfrac{1}{\sqrt{2}} \end{cases}$$

Which gives $\theta = \frac{\pi}{4}$ and $\phi = \frac{\pi}{4}$

(4) **Slopes**

$$\begin{aligned} r_\theta &= \langle -2\sin(\phi)\sin(\theta), 2\sin(\phi)\cos(\theta), 0\rangle \\ &= \left\langle -2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right), 2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right), 0\right\rangle \\ &\quad \text{(Use } \theta = \frac{\pi}{4}, \phi = \frac{\pi}{4}\text{)} \\ &= \langle -1, 1, 0\rangle \end{aligned}$$

$$\begin{aligned} r_\phi &= \langle 2\cos(\phi)\cos(\theta), 2\cos(\phi)\sin(\theta), -2\sin(\phi)\rangle \\ &= \left\langle 2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right), 2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right), -2\left(\frac{1}{\sqrt{2}}\right)\right\rangle \\ &= \left\langle 1, 1, -\sqrt{2}\right\rangle \end{aligned}$$

(5) **Normal Vector**

$$\begin{aligned}\hat{n} =& r_\theta \times r_\phi \\ =& \begin{vmatrix} i & j & k \\ -1 & 1 & 0 \\ 1 & 1 & -\sqrt{2} \end{vmatrix} \\ =& \left\langle -\sqrt{2}, -\sqrt{2}, -2 \right\rangle\end{aligned}$$

(6) **Point:** $(1, 1, \sqrt{2})$

(7) **Equation:**

$$-\sqrt{2}(x-1) - \sqrt{2}(y-1) - 2(z-\sqrt{2}) = 0$$