LECTURE 18: PARAMETRIC SURFACES (II)

Today we’ll do something unbelievable: We’ll calculate the area of any surface!

1. Quick Facts

Fact 1: Area of a parallelogram with sides \( \mathbf{a} \) and \( \mathbf{b} \):

\[ \| \mathbf{a} \times \mathbf{b} \| \]

Fact 2: If \( c, d > 0 \), then

\[ \| (c \mathbf{a}) \times (d \mathbf{b}) \| = \| \mathbf{a} \times \mathbf{b} \| cd \]

Date: Wednesday, February 19, 2020.
2. **Surface Areas**

**Goal:** Find the Area of a surface $S$:

**Strategy:**

1. Cover/Tile $S$ with mini parallelograms determined by tangent planes
2. Calculate the area $dS$ of each mini-parallelogram
3. Sum up/Integrate the mini areas

The result will be:

$$\text{Area } (S) = \int \int_{\text{Sum Mini-areas}} dS$$
What is \(dS\)?

**Recall:** Given \(r(u, v)\) (parametrization), \(r_u\) and \(r_v\) (partial derivatives) are on the tangent plane:

**STEP 1:** Consider the parallelogram with sides \((du) r_u\) and \((dv) r_v\):
STEP 2:

\[ dS = \text{Area(Parallelogram)} \]
\[ = \|(du)r_u \times (dv)r_v\| \text{ (By Fact 1)} \]
\[ = \|r_u \times r_v\| \, dudv \text{ (By Fact 2)} \]

STEP 3 (Integrate)

Surface Area (memorize)

\[
\text{Area (}S\text{)} = \int \int dS = \int \int_D \|r_u \times r_v\| \, dudv
\]

The rest of today is just examples of this formula.

3. EXAMPLES

Example 1:

Find Area(S), where \( S \) : Helicoid with equations

\[
r(u, v) = \langle u \cos(v), u \sin(v), v \rangle
\]

\[ 0 \leq u \leq 1, \, 0 \leq v \leq \pi \]

(1) Picture: (Taken from Wikipedia)
(2) \n\[ r_u = \langle \cos(v), \sin(v), 0 \rangle \]
\[ r_v = \langle -u \sin(v), u \cos(v), 1 \rangle \]

(3) \n\[ r_u \times r_v = \begin{vmatrix} i & j & k \\ \cos(v) & \sin(v) & 0 \\ -u \sin(v) & u \cos(v) & 1 \end{vmatrix} = \langle \sin(v), -\cos(v), u \cos^2(v) + u \sin^2(v) \rangle = \langle \sin(v), -\cos(v), u \rangle \]
(4) \[ dS = \|r_u \times r_v\| = \sqrt{\sin^2(v) + \cos^2(v) + u^2} = \sqrt{1 + u^2} \]

(5) \[
\text{Area (S)} = \iint dS \\
= \int_0^\pi \int_0^1 \sqrt{u^2 + 1} \, dudv \\
= \pi \int_0^1 \sqrt{u^2 + 1} \, du \\
= \cdots \text{ (Use the substitution} \ u = \tan(\theta)) \\
= \frac{\pi}{2} \left( \sqrt{2} + \ln(1 + \sqrt{2}) \right)
\]

Note: Know how to evaluate the following Math 2B integrals (click on the links for solutions)

(a) \[ \int \sqrt{x^2 + 1} \, dx \]

(b) \[ \int \sqrt{1 - x^2} \, dx \]

(c) \[ \int \sqrt{x^2 - 1} \, dx \]
OMG Example 2:
Find the surface area of a sphere of radius 1

Video: Surface Area of a Sphere

All those years you’ve been told that it’s $4\pi r^2$, but now I can finally show you why it’s true!

(1) Picture:

![Diagram of a sphere with parameters θ and φ]

(2) Parametrize: Basically spherical coordinates with $\rho = 1$

$$r(\theta, \phi) = \langle \sin(\phi) \cos(\theta), \sin(\phi) \sin(\theta), \cos(\phi) \rangle$$

$$0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi$$

(3)

$$r_\theta = \langle -\sin(\phi) \sin(\theta), \sin(\phi) \cos(\theta), 0 \rangle$$

$$r_\phi = \langle \cos(\phi) \cos(\theta), \cos(\phi) \sin(\theta), -\sin(\phi) \rangle$$
\( r_\theta \times r_\phi = \begin{vmatrix} i & j & k \\ -\sin(\phi)\sin(\theta) & \sin(\phi)\cos(\theta) & 0 \\ \cos(\phi)\cos(\theta) & \cos(\phi)\sin(\theta) & -\sin(\phi) \end{vmatrix} = \langle -\sin^2(\phi)\cos(\theta), -\sin^2(\phi)\sin(\theta), \\
-\sin(\phi)\cos(\phi)\sin^2(\theta) - \sin(\phi)\cos(\phi)\cos^2(\theta) \rangle \\
= \langle -\sin^2(\phi)\cos(\theta), -\sin^2(\phi)\sin(\theta), -\sin(\phi)\cos(\phi) \rangle \)

\( dS = ||r_\theta \times r_\phi|| \\
= (\sin^4(\phi)\cos^2(\theta) + \sin^4(\phi)\sin^2(\theta) + \sin^2(\phi)\cos^2(\phi))^\frac{1}{2} \\
= (\sin^4(\phi) + \sin^2(\phi)\cos^2(\phi))^\frac{1}{2} \\
= (\sin^2(\phi) (\sin^2(\theta) + \cos^2(\theta)))^\frac{1}{2} \\
= \sqrt{\sin^2(\phi)} \\
= \sin(\phi) \)

\( \text{Area (S)} = \int \int dS \\
= \int_0^\pi \int_0^{2\pi} \sin(\phi)d\theta d\phi \\
= 2\pi \left( \int_0^\pi \sin(\phi) d\phi \right) \\
= (2\pi)(2) \\
= 4\pi \\
= 4\pi (1)^2 \)
Note: In general, using the same method, you get that the surface area of a sphere of radius $r$ is $4\pi r^2$! WOW

4. The Case of Functions

What if $S$ is the graph of a function? In that case, the surface area formula becomes much simpler!

**Example 3:**

Find $\text{Area}(S)$, where $S$ is the portion of the paraboloid $z = 1 + x^2 + y^2$ over the disk $x^2 + y^2 = 4$

(1) Picture:

![Diagram of a paraboloid and a disk]

(2) $r(x, y) = \langle x, y, 1 + x^2 + y^2 \rangle$
(3)
\[ r_x = \langle 1, 0, 2x \rangle \]
\[ r_y = \langle 0, 1, 2y \rangle \]

(4)
\[ r_x \times r_y = \left| \begin{array}{ccc} i & j & k \\ 1 & 0 & 2x \\ 0 & 1 & 2y \end{array} \right| \]
\[ = \langle -2x, -2y, 1 \rangle \]
\[ = \left\langle -\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1 \right\rangle \]

(5)
\[ dS = \|r_x \times r_y\| = \sqrt{4x^2 + 4y^2 + 1} = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \]

(6)
Area \( S \) = \int \int dS
\[ = \int_0^{2\pi} \int_0^2 \sqrt{4x^2 + 4y^2 + 1} \, r \, dr \, d\theta \]
\[ = 2\pi \left[ \left(\frac{1}{8}\right) \left(\frac{2}{3}\right) (4r^2 + 1)^{3/2} \right]_0 \]
\[ = \frac{\pi}{6} (17^{3/2} - 1) \]
More Generally

If \( z = z(x, y) \), then

\[
\text{Area } (S) = \int \int_D \sqrt{1 + \left( \frac{\partial z}{\partial x} \right)^2 + \left( \frac{\partial z}{\partial y} \right)^2} \, dxdy
\]

**Note:** I personally think that it’s easier to do the problem the way I did it above instead of just memorizing the above formula, but in the end it’s up to you!

Example 4: (if time permits)

Area(\( S \)), where \( S \) : part of the surface \( z = xy \) inside the cylinder \( x^2 + y^2 = 1 \)

(1) **Picture:**

![Diagram of the surface and region](image)
Note: Could use \( r(x, y) = \langle x, y, xy \rangle \), but using the formula above instead, we get:

\[
\begin{align*}
\frac{\partial z}{\partial x} &= y, \\
\frac{\partial z}{\partial y} &= x
\end{align*}
\]

Area \((S)\) = \(\int \int_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \, dx \, dy\)

\[
= \int \int_D \sqrt{1 + y^2 + x^2} \, dx \, dy
\]

\[
= \int_0^{2\pi} \int_0^1 \left(\sqrt{1 + r^2}\right) r \, dr \, d\theta
\]

\[
= 2\pi \int_0^1 (r^2 + 1)^{\frac{1}{2}} r \, dr
\]

\[
= (2\pi) \left[ \left(\frac{1}{2}\right) \left(\frac{2}{3}\right) (r^2 + 1)^{\frac{3}{2}} \right]_0^1 (\text{Or use } u = r^2 + 1)
\]

\[
= \frac{2\pi}{3} \left( 2^{\frac{3}{2}} - 1 \right)
\]

\[
= \frac{2\pi}{3} \left( 2\sqrt{2} - 1 \right)
\]