

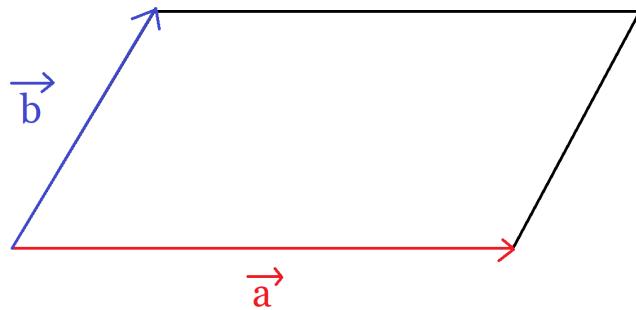
LECTURE 18: PARAMETRIC SURFACES (II)

Today we'll do something unbelievable: We'll calculate the area of **any** surface!

1. QUICK FACTS

Fact 1: Area of a parallelogram with sides \mathbf{a} and \mathbf{b} :

$$\|\mathbf{a} \times \mathbf{b}\|$$

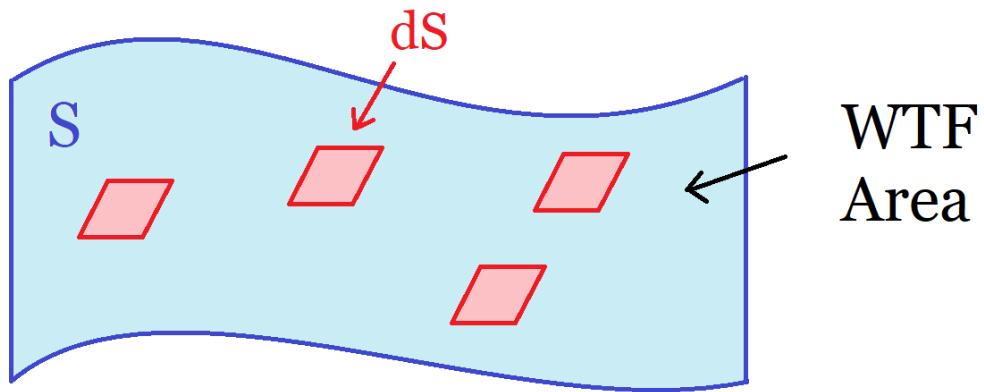


Fact 2: If $c, d > 0$, then

$$\|(c\mathbf{a}) \times (d\mathbf{b})\| = \|\mathbf{a} \times \mathbf{b}\| cd$$

2. SURFACE AREAS

Goal: Find the Area of a surface S :



Strategy:

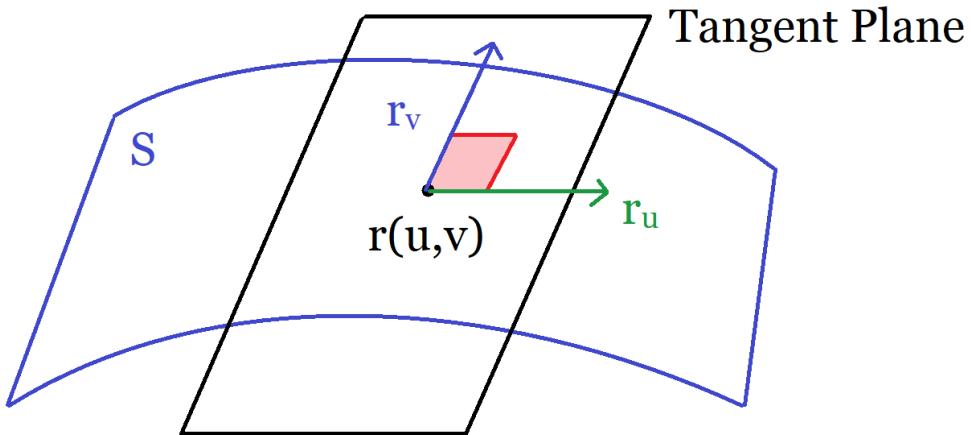
- (1) Cover/Tile S with mini parallelograms determined by tangent planes
- (2) Calculate the area dS of each mini-parallelogram
- (3) Sum up/Integrate the mini areas

The result will be:

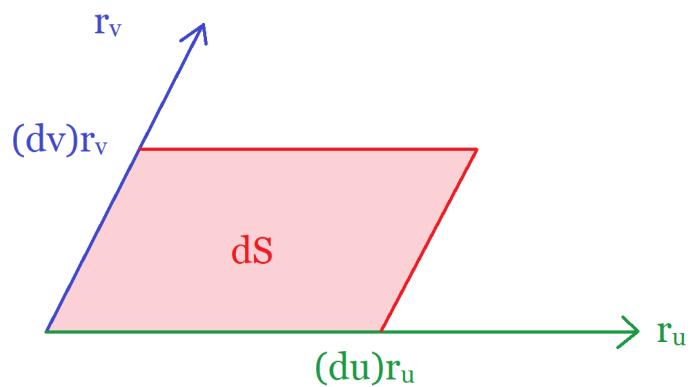
$$\text{Area } (S) = \underbrace{\iint}_{\text{Sum}} \underbrace{dS}_{\text{Mini-areas}}$$

What is dS ?

Recall: Given $r(u, v)$ (parametrization), r_u and r_v (partial derivatives) are on the tangent plane:



STEP 1: Consider the parallelogram with sides $\underbrace{(du)}_{\text{Small}} r_u$ and $\underbrace{(dv)}_{\text{Small}} r_v$:



STEP 2:

$$\begin{aligned}
 dS &= \text{Area(Parallelogram)} \\
 &= \|(du)r_u \times (dv)r_v\| \quad (\text{By Fact 1}) \\
 &= \|r_u \times r_v\| dudv \quad (\text{By Fact 2})
 \end{aligned}$$

STEP 3 (Integrate)**Surface Area (memorize)**

$$\text{Area } (S) = \int \int dS = \int \int_D \|r_u \times r_v\| dudv$$

The rest of today is just examples of this formula

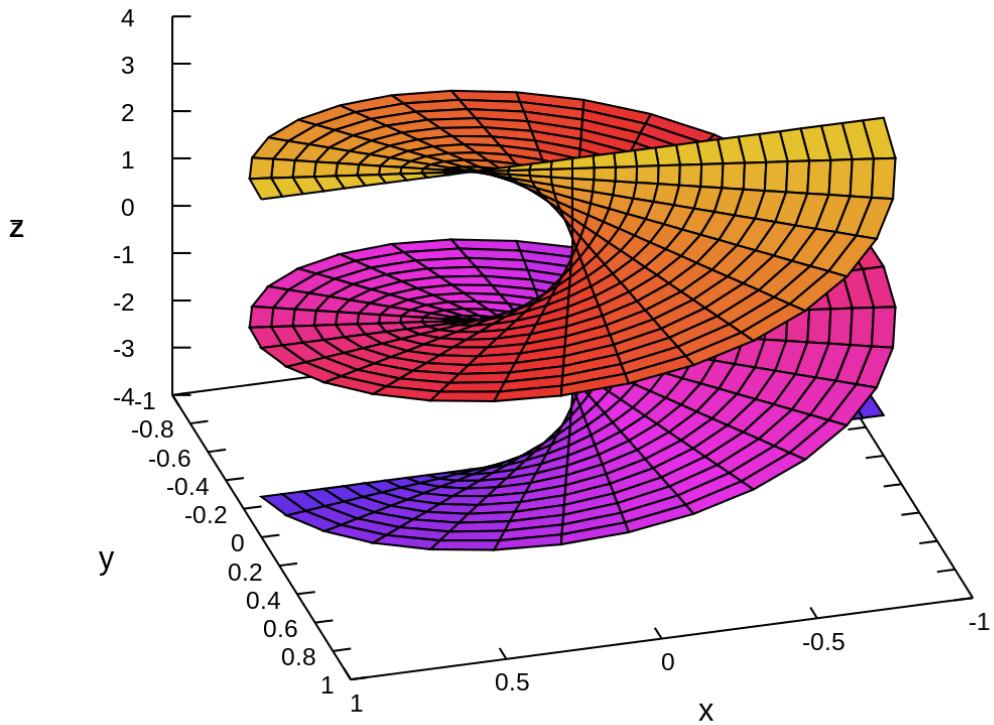
3. EXAMPLES**Example 1:**

Find Area(S), where S : Helicoid with equations

$$r(u, v) = \langle u \cos(v), u \sin(v), v \rangle$$

$$0 \leq u \leq 1, 0 \leq v \leq \pi$$

(1) **Picture:** (Taken from Wikipedia)



(2)

$$r_u = \langle \cos(v), \sin(v), 0 \rangle$$

$$r_v = \langle -u \sin(v), u \cos(v), 1 \rangle$$

(3)

$$\begin{aligned}
 r_u \times r_v &= \begin{vmatrix} i & j & k \\ \cos(v) & \sin(v) & 0 \\ -u \sin(v) & u \cos(v) & 1 \end{vmatrix} \\
 &= \langle \sin(v), -\cos(v), u \cos^2(v) + u \sin^2(v) \rangle \\
 &= \langle \sin(v), -\cos(v), u \rangle
 \end{aligned}$$

(4)

$$dS = \|r_u \times r_v\| = \sqrt{\sin^2(v) + \cos^2(v) + u^2} = \sqrt{1 + u^2}$$

(5)

$$\begin{aligned}\text{Area } (S) &= \int \int dS \\ &= \int_0^\pi \int_0^1 \sqrt{u^2 + 1} du dv \\ &= \pi \int_0^1 \sqrt{u^2 + 1} du \\ &= \dots (\text{ Use the substitution } u = \tan(\theta)) \\ &= \frac{\pi}{2} \left(\sqrt{2} + \ln(1 + \sqrt{2}) \right)\end{aligned}$$

Note: Know how to evaluate the following Math 2B integrals (click on the links for solutions)

$$(a) \int \sqrt{x^2 + 1} dx$$

$$(b) \int \sqrt{1 - x^2} dx$$

$$(c) \int \sqrt{x^2 - 1} dx$$

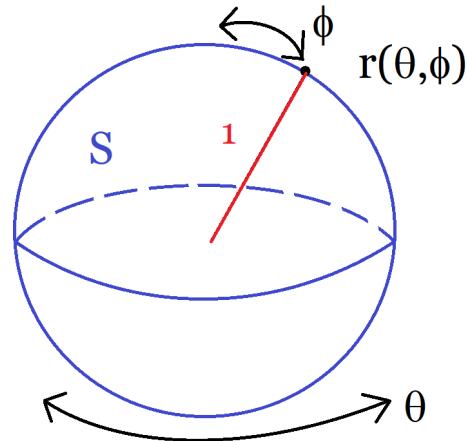
OMG Example 2:

Find the surface area of a sphere of radius 1

Video: Surface Area of a Sphere

All those years you've been told that it's $4\pi r^2$, but now I can finally show you why it's true!

(1) **Picture:**



(2) **Parametrize:** Basically spherical coordinates with $\rho = 1$

$$r(\theta, \phi) = \langle \sin(\phi) \cos(\theta), \sin(\phi) \sin(\theta), \cos(\phi) \rangle$$

$$0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi$$

(3)

$$r_\theta = \langle -\sin(\phi) \sin(\theta), \sin(\phi) \cos(\theta), 0 \rangle$$

$$r_\phi = \langle \cos(\phi) \cos(\theta), \cos(\phi) \sin(\theta), -\sin(\phi) \rangle$$

(4)

$$\begin{aligned}
r_\theta \times r_\phi &= \begin{vmatrix} i & j & k \\ -\sin(\phi) \sin(\theta) & \sin(\phi) \cos(\theta) & 0 \\ \cos(\phi) \cos(\theta) & \cos(\phi) \sin(\theta) & -\sin(\phi) \end{vmatrix} \\
&= \langle -\sin^2(\phi) \cos(\theta), -\sin^2(\phi) \sin(\theta), \\
&\quad -\sin(\phi) \cos(\phi) \textcolor{blue}{\sin^2(\theta)} - \sin(\phi) \cos(\phi) \textcolor{blue}{\cos^2(\theta)} \rangle \\
&= \langle -\sin^2(\phi) \cos(\theta), -\sin^2(\phi) \sin(\theta), -\sin(\phi) \cos(\phi) \rangle
\end{aligned}$$

(5)

$$\begin{aligned}
dS &= \|r_\theta \times r_\phi\| \\
&= (\sin^4(\phi) \textcolor{blue}{\cos^2(\theta)} + \sin^4(\phi) \textcolor{blue}{\sin^2(\theta)} + \sin^2(\phi) \cos^2(\phi))^{\frac{1}{2}} \\
&= (\sin^4(\phi) + \sin^2(\phi) \cos^2(\phi))^{\frac{1}{2}} \\
&= (\sin^2(\phi) (\sin^2(\theta) + \cos^2(\theta)))^{\frac{1}{2}} \\
&= \left(\sqrt{\sin^2(\phi)} \right) \\
&= \sin(\phi)
\end{aligned}$$

(6)

$$\begin{aligned}
\text{Area } (S) &= \int \int dS \\
&= \int_0^\pi \int_0^{2\pi} \sin(\phi) d\theta d\phi \\
&= 2\pi \left(\int_0^\pi \sin(\phi) d\phi \right) \\
&= (2\pi)(2) \\
&= 4\pi \\
&= 4\pi(1)^2
\end{aligned}$$

Note: In general, using the same method, you get that the surface area of a sphere of radius r is $4\pi r^2$! **WOW**

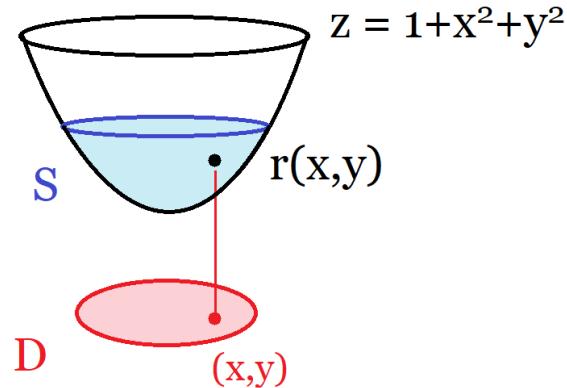
4. THE CASE OF FUNCTIONS

What if S is the graph of a function? In that case, the surface area formula becomes much simpler!

Example 3:

Find Area(S), where S is the portion of the paraboloid $z = 1 + x^2 + y^2$ over the disk $x^2 + y^2 \leq 4$

(1) Picture:



$$(2) \quad r(x, y) = \langle x, y, 1 + x^2 + y^2 \rangle$$

(3)

$$\begin{aligned} r_x &= \langle 1, 0, 2x \rangle \\ r_y &= \langle 0, 1, 2y \rangle \end{aligned}$$

(4)

$$\begin{aligned} r_x \times r_y &= \begin{vmatrix} i & j & k \\ 1 & 0 & 2x \\ 0 & 1 & 2y \end{vmatrix} \\ &= \langle -2x, -2y, 1 \rangle \\ &= \left\langle -\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1 \right\rangle \end{aligned}$$

(5)

$$dS = \|r_x \times r_y\| = \sqrt{4x^2 + 4y^2 + 1} = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}$$

(6)

$$\begin{aligned} \text{Area } (S) &= \int \int dS \\ &= \int \int_D \sqrt{4x^2 + 4y^2 + 1} dx dy \\ &= \int_0^{2\pi} \int_0^2 (4r^2 + 1)^{\frac{1}{2}} r dr d\theta \\ &= 2\pi \left[\left(\frac{1}{8}\right) \left(\frac{2}{3}\right) (4r^2 + 1)^{\frac{3}{2}} \right]_0^2 \quad (\text{Or use } u = 4r^2 + 1) \\ &= \frac{\pi}{6} \left((17)^{\frac{3}{2}} - 1 \right) \end{aligned}$$

More Generally

If $z = z(x, y)$, then

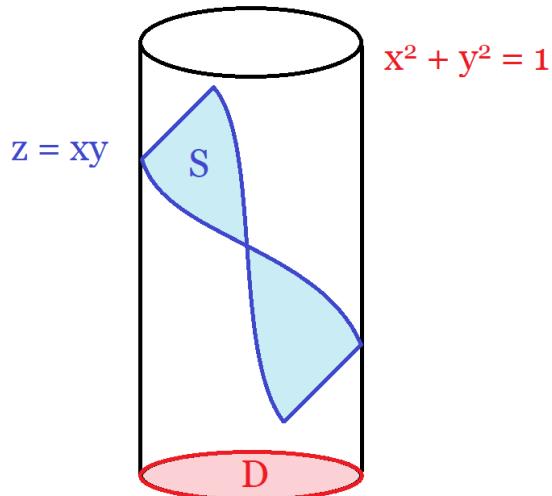
$$\text{Area } (S) = \int \int_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dx dy$$

Note: I personally think that it's easier to do the problem the way I did it above instead of just memorizing the above formula, but in the end it's up to you!

Example 4: (if time permits)

Area(S), where S : part of the surface $z = xy$ inside the cylinder $x^2 + y^2 = 1$

(1) Picture:



(2) **Note:** Could use $r(x, y) = \langle x, y, xy \rangle$, but using the formula above instead, we get:

$$\frac{\partial z}{\partial x} = y, \quad \frac{\partial z}{\partial y} = x$$

$$\begin{aligned} \text{Area } (S) &= \int \int_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dx dy \\ &= \int \int_D \sqrt{1 + y^2 + x^2} dx dy \\ &= \int_0^{2\pi} \int_0^1 \left(\sqrt{1 + r^2}\right) r dr d\theta \\ &= 2\pi \int_0^1 (r^2 + 1)^{\frac{1}{2}} r dr \\ &= (2\pi) \left[\left(\frac{1}{2}\right) \left(\frac{2}{3}\right) (r^2 + 1)^{\frac{3}{2}} \right]_0^1 \text{ (Or use } u = r^2 + 1) \\ &= \frac{2\pi}{3} \left(2^{\frac{3}{2}} - 1\right) \\ &= \frac{2\pi}{3} (2\sqrt{2} - 1) \end{aligned}$$