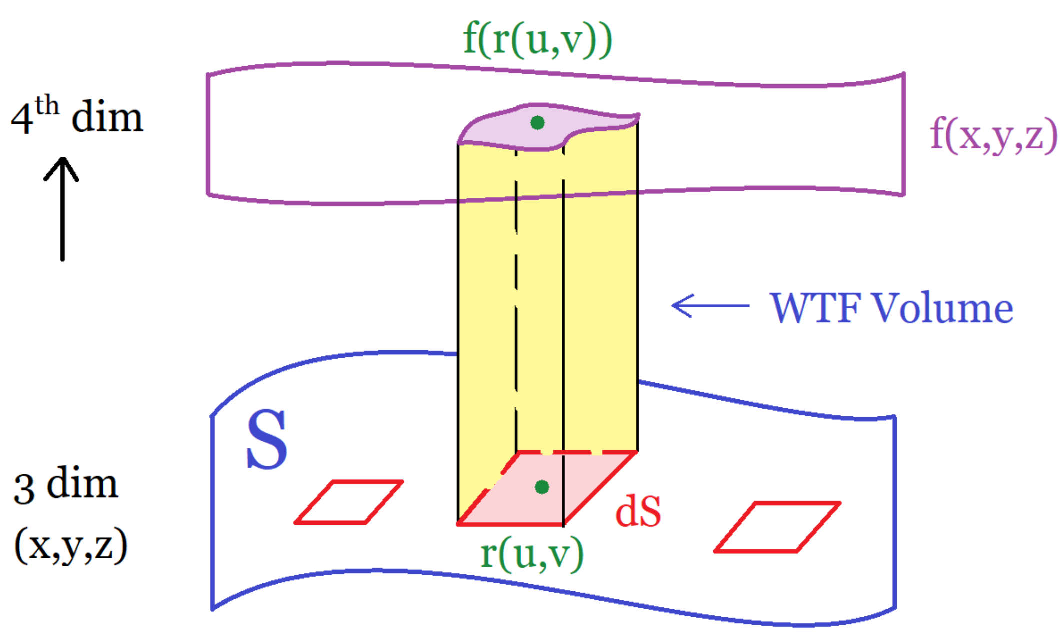
LECTURE 19: SURFACE INTEGRALS (I)

1. Quick Facts

Today: We're going to integrate a function over a surface (WOW!)

Goal: Find the volume of the solid under a function f and over a surface S.

Picture:



Date: Friday, February 21, 2020.

Note: The solid itself lives in the 4th dimension, but it is 3-dimensional because S has dimension 2 and the height has dimension 1.

Here is where our mini-parallelograms dS help us once again!

Recall

$$dS = \text{Area of mini parallelogram} = \|r_u \times r_v\|\, dudv$$

Idea:

Height $= f$, Base $= dS \Rightarrow$ Mini-Volume $=$ Height $\times$ Base $= f\, dS$.

And sum ($=$ integrate) everything up to get:

Surface Integral

$$\int\int_S f dS = \int\int_D f(r(u,v)) \underbrace{\|r_u \times r_v\|\, dudv}_{dS}$$

Mnemonic: Sum ($= \int\int$) of Height ($= f$) times Base ($dS = \|r_u \times r_v\|$)

2. Examples

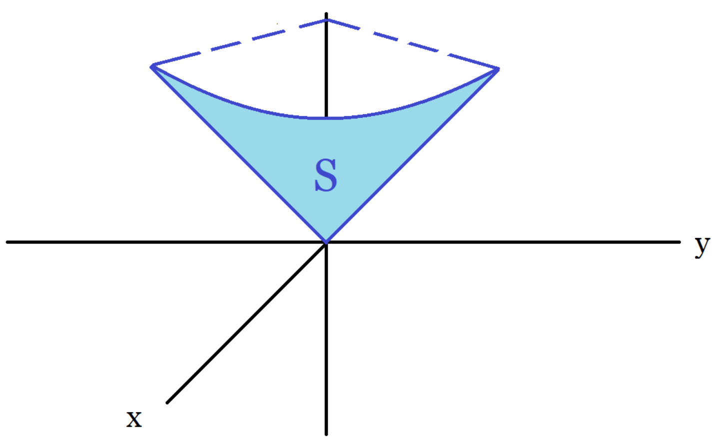
Example 1:

$$\int\int_S y dS$$

S : Cone in the first octant, parametrized by:

$$r(u, v) = \langle u\cos(v), u\sin(v), u\rangle \, 0 \leq u \leq 1, 0 \leq v \leq \frac{\pi}{2}$$

(1) **Picture:**



(2) **Slopes**

$$r_u = \langle \cos(v), \sin(v), 1\rangle$$
$$r_v = \langle -u\sin(v), u\cos(v), 0\rangle$$

(3) **Normal Vector**

$$
\begin{aligned}
r_u \times r_v &= \begin{vmatrix} i & j & k \\ \cos(v) & \sin(v) & 1 \\ -u\sin(v) & u\cos(v) & 0 \end{vmatrix} \\
&= \left\langle -u\cos(v), -u\sin(v), u\cos^2(v) + u\sin^2(v) \right\rangle \\
&= \langle -u\cos(v), -u\sin(v), u \rangle
\end{aligned}
$$

(4) **dS**

$$
\begin{aligned}
dS &= \|r_u \times r_v\| \\
&= \left(u^2\cos^2(v) + u^2\sin^2(v) + u^2\right)^{\frac{1}{2}} \\
&= \sqrt{u^2 + u^2} \\
&= \sqrt{2}u
\end{aligned}
$$

(5)

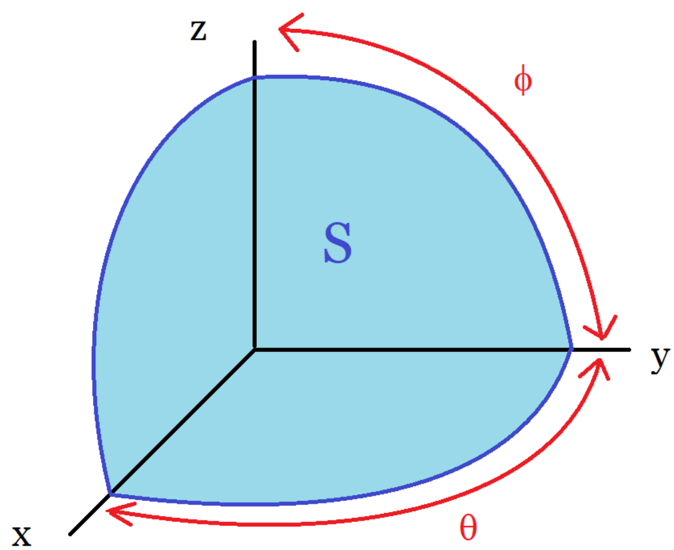
$$
\begin{aligned}
\int\int_S y\, dS &= \int\int_D u\sin(v)\, \|r_u \times r_v\|\, dudv \\
&= \int_0^{\frac{\pi}{2}} \int_0^1 u\sin(v)\, \sqrt{2}ududv \\
&= \left(\int_0^1 \sqrt{2}u^2 du\right)\left(\int_0^{\frac{\pi}{2}} \sin(v)dv\right) \\
&= \frac{\sqrt{2}}{3}
\end{aligned}
$$

Example 2:

$$\int\int_S z dS$$

S : Sphere of radius 1 in the first octant

(1) **Picture:**



(2) **Parametrize:** Spherical with $\rho = 1$:

$$r(\theta, \phi) = \langle \sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi)\rangle \qquad 0 \leq \theta \leq \frac{\pi}{2}, 0 \leq \phi \leq \frac{\pi}{2}$$

(3) **Slopes and dS:** Last time (See OMG Example 2 of Lecture 18), found:

$$dS = \|r_\theta \times r_\phi\| = \sin(\phi)$$

(4)

$$\begin{aligned}\int\int_S z\, dS &= \int_0^{\frac{\pi}{2}} \cos(\phi)\sin(\phi)d\theta d\phi \\ &= \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \cos(\phi)\sin(\phi)d\phi \\ &= (\text{Use } u = \sin(\phi)) \\ &= \frac{\pi}{4}\end{aligned}$$

Interpretations:

(1) $\int\int f dS =$ Volume under f and over S

(2) If $f =$ Density, then $\int\int f dS =$ Mass of S (think mass of a metal plate)

(3) $Area(S) = \int\int_S dS$ (last time) $= \int\int 1 dS$ (so this is a generalization of surface area from last time)

3. The case of Functions

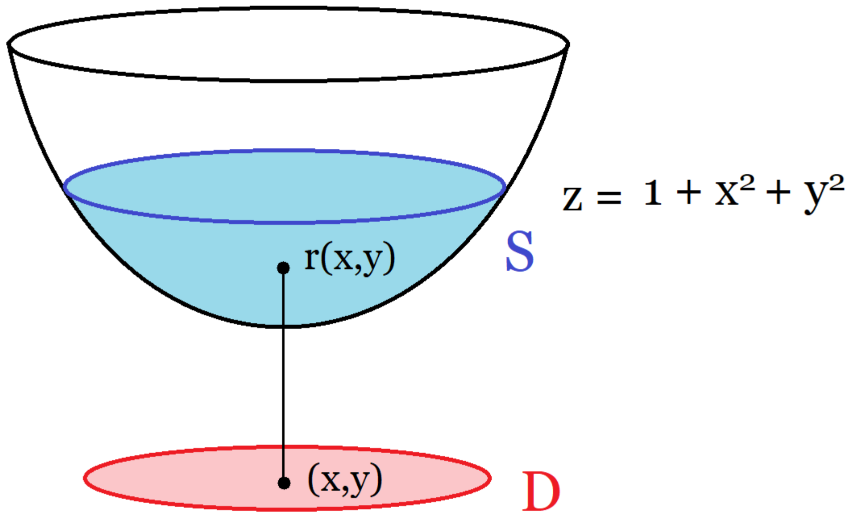
Just like last time, if S is the graph of a function, then the formula for $\int\int_S f dS$ simplifies a bit:

Example 3:

$$\int\int_S \sqrt{z + 3x^2 + 3y^2} dS$$

S : Graph of $z = 1 + x^2 + y^2$ over the disk of radius 3

(1) **Picture:**



Warning: Here $z = 1+x^2+y^2$ is our surface S, but $f(x, y, z) = \sqrt{z + 3x^2 + 3y^2}$ is the function we want to integrate. Do not confuse the two!

(2) **Parametrize**

$$r(x, y) = \langle x, y, 1 + x^2 + y^2 \rangle$$

(3) **Slopes**

$$\begin{aligned} r_x &= \langle 1, 0, 2x\rangle \\ r_y &= \langle 0, 1, 2y\rangle \end{aligned}$$

(4) **Normal Vector**

$$r_x \times r_y = \begin{vmatrix} i & j & k \\ 1 & 0 & 2x \\ 0 & 1 & 2y \end{vmatrix} = \langle -2x, -2y, 1\rangle$$

(5) dS

$$dS = \|r_x \times r_y\| = \sqrt{4x^2 + 4y^2 + 1} = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}$$

(6)

$$\begin{aligned}
\int\int_S \sqrt{z + 3x^2 + 3y^2}\, dS &= \int\int_D \left(1 + x^2 + y^2 + 3x^2 + 3y^2\right)^{\frac{1}{2}} \sqrt{4x^2 + 4y^2 + 1} dxdy \\
&= \int\int_D \sqrt{4x^2 + 4y^2 + 1}\sqrt{4x^2 + 4y^2 + 1} dxdy \\
&= \int\int_D 4x^2 + 4y^2 + 1 dxdy \\
&= \int_0^{2\pi} \int_0^3 (4r^2 + 1) r dr d\theta \\
&= 171\pi
\end{aligned}$$

More Generally:

If $z = z(x, y)$, then

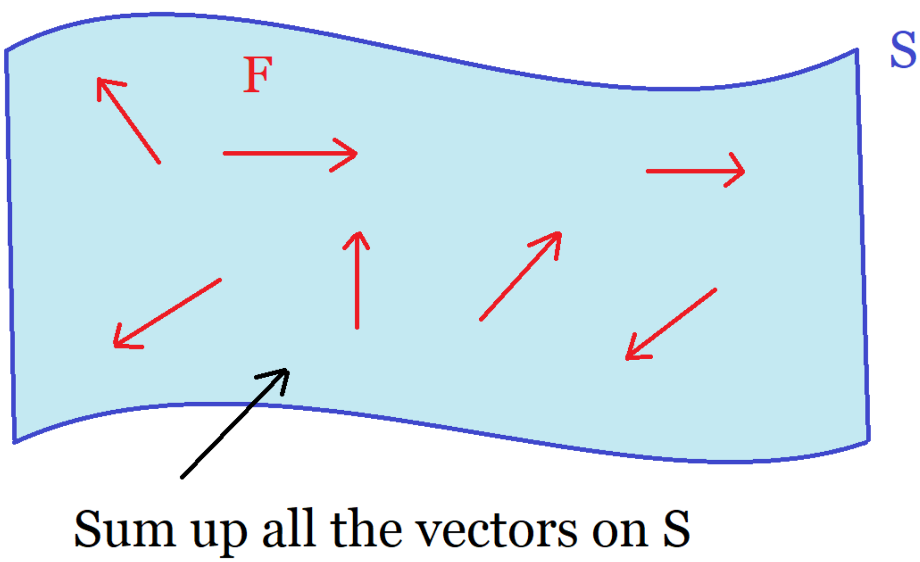
$$\int\int_S f(x, y, z) dS = \int\int_D f(x, y, z(x, y)) \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}$$

DON'T memorize this, just do the problem the way I did above!

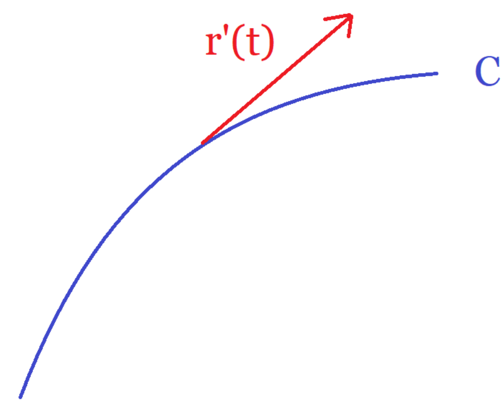
4. Surface Integrals of Vector Fields

Back to vector fields!

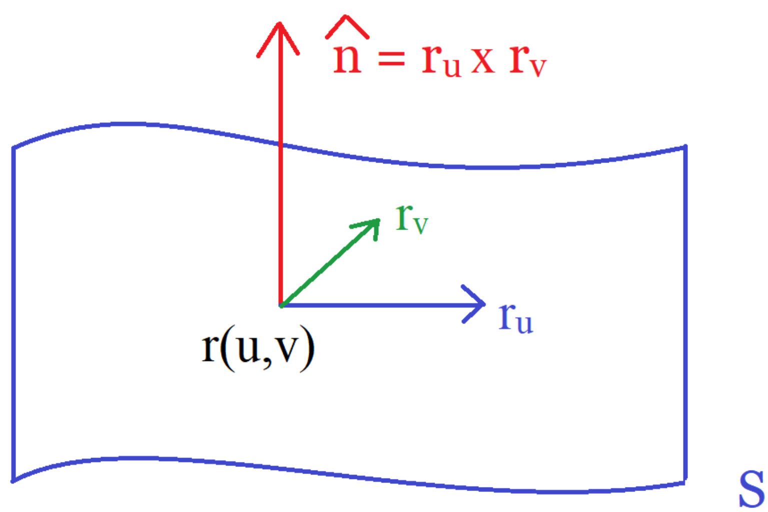
Goal: Given a vector field F and a surface S, want to sum up the values of F over S (think like collecting all the vectors F on a magic carpet S)



Note: Before had $F \cdot dr = F \cdot r'(t)dt$, where $r'(t)$ is the **direction** vector of C.



What is the analog of $r'(t)$ for a surface? The NORMAL VECTOR $\hat{n} = r_u \times r_v$!!!



Why $\hat{n}$?

(1) What the direction vector is to a line, the normal vector is to a plane

(2) A plane has many direction vectors, but only one normal vector (up to a $\pm$ sign)

(3) $\hat{n}$ is a vector that involves derivatives of r

To get the surface integral, you just dot F with the normal vector $\hat{n}$:

Surface Integral of F

$$\int\int_S F \cdot d\mathbf{S} = \int\int F \cdot \hat{n} = \int\int_D F(r(u,v)) \cdot \underbrace{(r_u \times r_v)}_{\hat{n}} dudv$$

Note: Compare to

$$\int_C F \cdot dr = \int F(r(t)) \cdot r'(t) dt$$

In both cases, you dot F with something that involves derivatives

WARNING:

Don't confuse $\int\int_S f dS$ (surface integral of a function) with $\int\int_S F \cdot d\mathbf{S}$ (surface integral of a vector field). Two completely different topics (for now)

(Will do many examples next time)