LECTURE 21: THE DIVERGENCE THEOREM (I)

Welcome to the third FTC for vector fields. In my opinion it’s the most powerful one because it simplifies your work tremendously! For this we need to define a new operation related to vector fields:

1. DIVERGENCE

<table>
<thead>
<tr>
<th>Divergence</th>
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<tbody>
<tr>
<td>If $F = \langle P, Q, R \rangle$, then $\text{div}(F) = P_x + Q_y + R_z$</td>
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Example: $F = \langle x^2, y^2, z^2 \rangle$

$$\text{div}(F) = (x^2)_x + (y^2)_y + (z^2)_z = 2x + 2y + 2z$$

Example: $F = \langle \tan^{-1}(xz), e^{yz}, \ln(1 + xz) \rangle$

$$\text{div}(F) = (\tan^{-1}(xz))_x + (e^{yz})_y + (\ln(1 + xz))_z$$

$$= \frac{1}{(xz)^2} + z + e^{yz}z + \frac{1}{1 + xz}x$$

Interpretation: $\text{div}(F)$ measures the expansion of $F$

Example: $\text{div}(\langle x, y, z \rangle) = 1 + 1 + 1 = 3$

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$F$ “expands” at a rate of 3

In fact: If $\text{div}(F) = 0$, then $F$ is called *incompressible* (= non-expanding)

2. **The Divergence Theorem**

Motivation:

2B: $\int \int F = \int \int \int F'$

The Divergence Theorem

$$\int \int_S F \cdot dS = \int \int \int_E \text{div}(F) dx dy dz$$
Remarks:

(1) Here $S$ is a closed surface, and $E$ is the region inside $S$.

(2) Awesome, because it converts a surface integral (HARD) into a triple integral (EASY).

(3) Compare with Green’s Theorem: $\int_C F \cdot dr = \int \int_D Q_x - P_y$. The Div Theorem is really a 3D version of Green, because Green converts a line integral into a double integral, but this one converts a surface integral into a triple integral.

(4) **Interpretation:** If you add up all the mini-expansions $\text{div}(F)$ over $E$, you get the net flux of $F$ over $S$: 
(5) Important: $\hat{n}$ has to point outwards

3. Examples

Video: [The Divergence Theorem](#)

**Example 1:**

\[
\int \int_S F \cdot dS
\]

\[F = \langle 3x, 2y, -z \rangle, \ S : \text{Sphere of Radius 2}\]

(1) Picture:
LECTURE 21: THE DIVERGENCE THEOREM (I)

(2)

\[ \int \int_S F \cdot dS = \int \int \int_E \text{div}(F)\,dxdydz \]

\[ (3x)_x + (2y)_y + (-z)_z = 3 + 2 - 1 = 4 \]

\[ = \int \int \int_E 4dxdydz \]

\[ = 4\text{Vol}(E) \]

\[ = 4 \left( \frac{4\pi 2^3}{3} \right) \]

\[ = \frac{128\pi}{3} \quad \text{WOW! Effortless!} \]

\textbf{Example 2:}

\[ \int \int_S F \cdot dS \]

\[ F = \left\langle xy, y^2 + e^{xz^2}, \sin(xy) \right\rangle \]

\(S\) : Surface of the region bounded by \(x^2 + y^2 = 1, z = 0\), and \(y + z = 2\)
(1) **Picture:**

![Diagram of a cylindrical region with equations for the surfaces and a surface S.]

(2) **Note:** Evaluating $\int \int_S F \cdot d\mathbf{S}$ directly is **painful**, you would have to evaluate 3 different surface integrals!

\[
\int \int_S F \cdot d\mathbf{S} = \int \int \int_E \text{div}(F) \, dx\, dy\, dz \\
= \int \int \int_E (xy)_x + (y^2 + e^{xz^2})_y + (\sin(xy))_z \\
= \int \int \int_E y + 2y + 0 \\
= \int \int \int_E 3y \, dx\, dy\, dz
\]
(3) **Inequalities:**

\[
0 \leq z \leq 2 - y \\
0 \leq z \leq 2 - r \sin(\theta)
\]

\[
0 \leq r \leq 1 \\
0 \leq \theta \leq 2\pi
\]

(4)

\[
= \int_0^{2\pi} \int_0^1 \int_0^{2-r \sin(\theta)} 3r \sin(\theta) \, r dz dr d\theta
\]

\[
= \cdots
\]

\[
= - \frac{3\pi}{4}
\]

4. **Closing a Surface**

**Example 3: (Tricky!)**

\[
\int \int_S F \cdot dS
\]

\[
F = \left\langle z^2 x, \frac{1}{3} y^3 + \tan(z), x^2 z + y^2 \right\rangle
\]

\(S:\) Top half of sphere \(x^2 + y^2 + z^2 = 1\)

(STEP 1) **Picture:**
WARNING: $S$ is not closed! (doesn’t include the bottom lid), so need to close it!

Let $S' = \text{bottom disk}$, then $S + S'$ (= Top Sphere + Bottom Disk) is closed, so by the Divergence Theorem:
\[ \int \int_{S+S'} F \cdot dS = \int \int \int_E \text{div}(F) \, dx \, dy \, dz \]

\[ = \int \int \int_E \left( z^2 x_x + \left( \frac{1}{3} y^3 + \tan(z) \right)_y + (x^2 z + y^2)_z \right) \, dx \, dy \, dz \]

\[ = \int \int \int_E z^2 + y^2 + x^2 \, dx \, dy \, dz \]

\[ = \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^1 \rho^2 \rho^2 \sin(\phi) \, d\rho \, d\theta \, d\phi \]

\[ = 2\pi \left( \int_0^{\frac{\pi}{2}} \sin(\phi) \, d\phi \right) \left( \int_0^1 \rho^4 \, d\rho \right) \]

\[ = \frac{2\pi}{5} \]
(STEP 2)

\[ \int \int_{S + S'} F \cdot dS = \frac{2\pi}{5} \]

\[ \int \int_{S} F \cdot dS + \int \int_{S'} F \cdot dS = \frac{2\pi}{5} \]

\[ \int \int_{S} F \cdot dS = \frac{2\pi}{5} - \int \int_{S'} F \cdot dS \]

(STEP 3) \( \int \int_{S'} F \cdot dS \)

**WARNING:** Outward orientation takes precedence over upward orientation here!

So make sure that \( \hat{n} \) points **downwards** here!
(1) Parametrize
\[ r(x, y) = \langle x, y, 0 \rangle \] (or use polar coordinates, that’s ok too)

(2) Slopes
\[ r_x = \langle 1, 0, 0 \rangle \]
\[ r_y = \langle 0, 1, 0 \rangle \]

(3) Normal Vector
\[ \hat{n} = r_x \times r_y = \left| \begin{array}{ccc} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right| = \langle 0, 0, 1 \rangle \]

Since we want \( \hat{n} \) to point downwards, we choose \( \hat{n} = \langle 0, 0, -1 \rangle \)

(4)
\[
\int \int_{S'} F \cdot dS = \int \int_D F \cdot \hat{n} \, dx \, dy = \int \int_D \left\langle 0x^2 + \frac{1}{3}y^3 + \tan(0), x^2(0) + y^2 \right\rangle \cdot \langle 0, 0, -1 \rangle \, dx \, dy \\
= \int \int_D -y^2 \, dx \, dy \quad D: \text{ Disk of radius } 1 \\
= \int \int_D -r^2 \sin^2(\theta) \, r \, dr \, d\theta \\
= \cdots \\
= -\frac{\pi}{4}
\]
(STEP 4) **Answer:** From STEP 2, we have:

\[
\int \int_S F \cdot dS = \frac{2\pi}{5} - \int \int_{S'} F \cdot dS
\]

\[
= \frac{2\pi}{5} - \left( -\frac{\pi}{4} \right)
\]

\[
= \frac{13\pi}{20}
\]