LECTURE 21: THE DIVERGENCE THEOREM (I)

Welcome to the *third* FTC for vector fields. In my opinion it's the most powerful one because it simplifies your work tremendously! For this we need to define a new operation related to vector fields:

1. Divergence

Divergence

If $F = \langle P, Q, R \rangle$, then

$$\mathrm{div}(F) = P_x + Q_y + R_z$$

Example: $F = \left\langle x^2, y^2, z^2 \right\rangle$

$$\mathrm{div}(F) = \left(x^2\right)_x + \left(y^2\right)_y + \left(z^2\right)_z = \underbrace{2x + 2y + 2z}_{\text{A Number}}$$

Example: $F = \left\langle \tan^{-1}(xz), e^{yz}, \ln(1+xz) \right\rangle$

$$\begin{aligned}\mathrm{div}(F) &= \left(\tan^{-1}(xz)\right)_x + \left(e^{yz}\right)_y + \left(\ln(1+xz)\right)_z \\ &= \frac{1}{(xz)^2+1} z + e^{yz} z + \frac{1}{1+xz} x\end{aligned}$$

Interpretation: $\mathrm{div}(F)$ measures the **expansion** of F

Example: $\mathrm{div}(\langle x, y, z \rangle) = 1 + 1 + 1 = 3$

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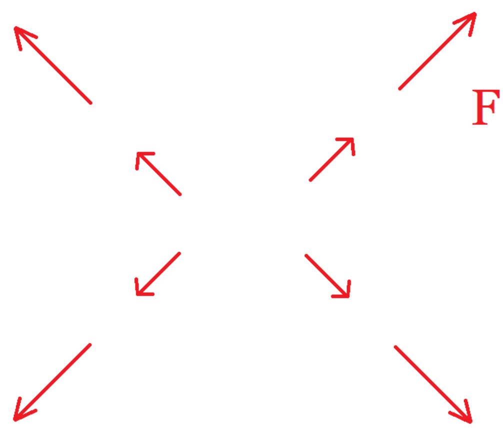


F "expands" at a rate of 3

In fact: If $\text{div}(F) = 0$, then F is called *incompressible* (= non-expanding)

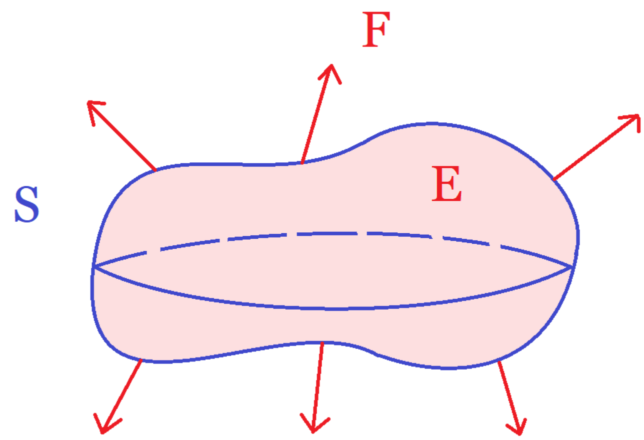
2. The Divergence Theorem

Motivation:

$$\text{2B:} \qquad \int\int F = \int\int\int F'$$

The Divergence Theorem

$$\int\int_S F \cdot d\mathbf{S} = \int\int\int_E \text{div}(F) dxdydz$$



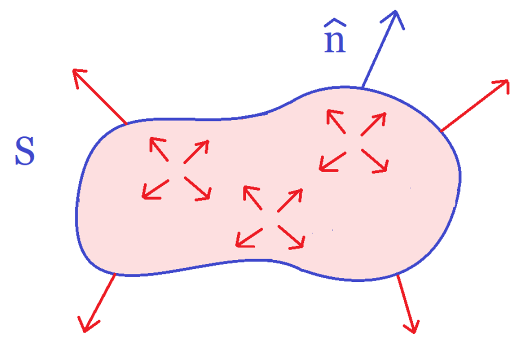
Remarks:

(1) Here S is a closed surface, and E is the region inside S

(2) Awesome, because it converts a surface integral (HARD) into a triple integral (EASY)

(3) Compare with Green's Theorem: $\int_C F \cdot dr = \int\int_D Q_x - P_y$. The Div Theorem is really a 3D version of Green, because Green converts a line integral into a double integral, but this one coverts a surface integral into a triple integral.

(4) **Interpretation:** If you add up all the mini-expansions $\mathrm{div}(F)$ over E, you get the net flux of F over S:



(5) Important: $\hat{n}$ has to point **outwards**

3. Examples

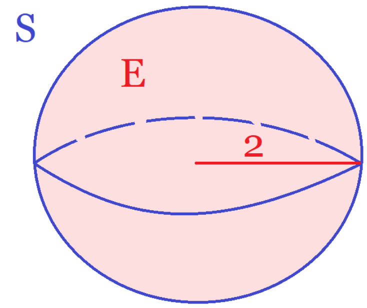
Video: The Divergence Theorem

Example 1:

$$\int\int_S F \cdot d\mathbf{S}$$

$F = \langle 3x, 2y, -z \rangle$, S : Sphere of Radius 2

(1) **Picture:**



(2)

$$\int\int_S F \cdot d\mathbf{S} = \int\int\int_E \operatorname{div}(F) dxdydz$$

$$(3x)_x + (2y)_y + (-z)_z = 3 + 2 - 1 = 4$$

$$= \int\int\int_E 4dxdydz$$

$$=4Vol(E)$$

$$=4\left(\frac{4}{3}\pi 2^3\right)$$

$$=\frac{128\pi}{3}$$

$WOW!$ Effortless!

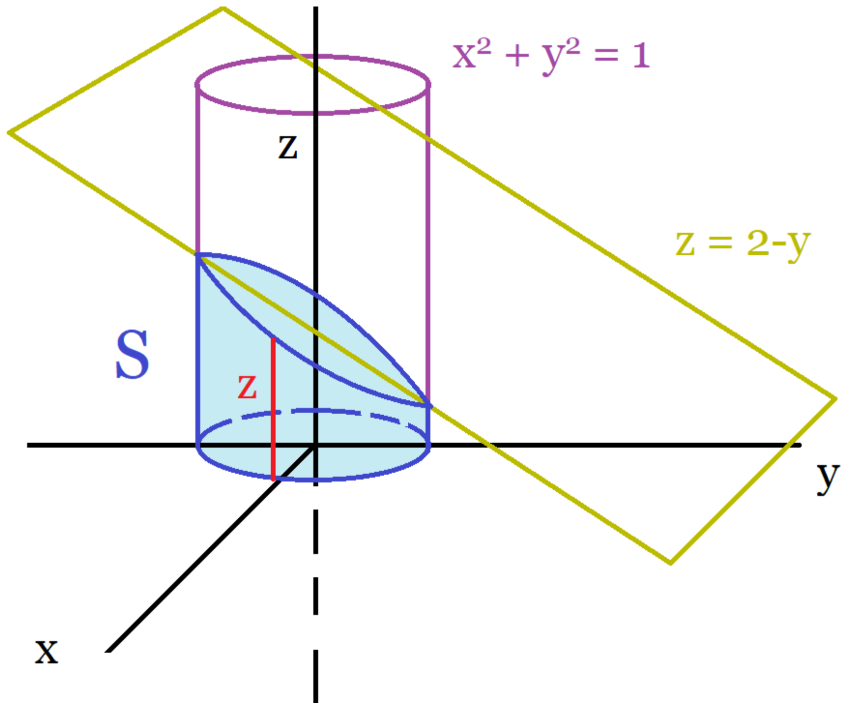
Example 2:

$$\int\int_S F \cdot d\mathbf{S}$$

$F = \left\langle xy, y^2 + e^{xz^2}, \sin(xy)\right\rangle$

S : Surface of the region bounded by $x^2 + y^2 = 1, z = 0$, and $y + z = 2$

(1) **Picture:**



(2) **Note:** Evaluating $\int\int_S F \cdot d\mathbf{S}$ directly is **painful**, you would have to evaluate 3 different surface integrals!

$$\int\int_S F \cdot d\mathbf{S} = \int\int\int_E \mathrm{div}(F) dxdydz$$
$$= \int\int\int_E (xy)_x + \left(y^2 + e^{xz^2}\right)_y + (\sin(xy))_z$$
$$= \int\int\int_E y + 2y + 0$$
$$= \int\int\int_E 3ydxdydz$$

(3) **Inequalities:**

$$0 \leq z \leq 2 - y$$
$$0 \leq z \leq 2 - r\sin(\theta)$$

$$0 \leq r \leq 1$$
$$0 \leq \theta \leq 2\pi$$

(4)

$$= \int_0^{2\pi} \int_0^1 \int_0^{2-r\sin(\theta)} \underbrace{3r\sin(\theta)}_{3y}\, r dz dr d\theta$$
$$= \cdots$$
$$= -\frac{3\pi}{4}$$

4. Closing a Surface

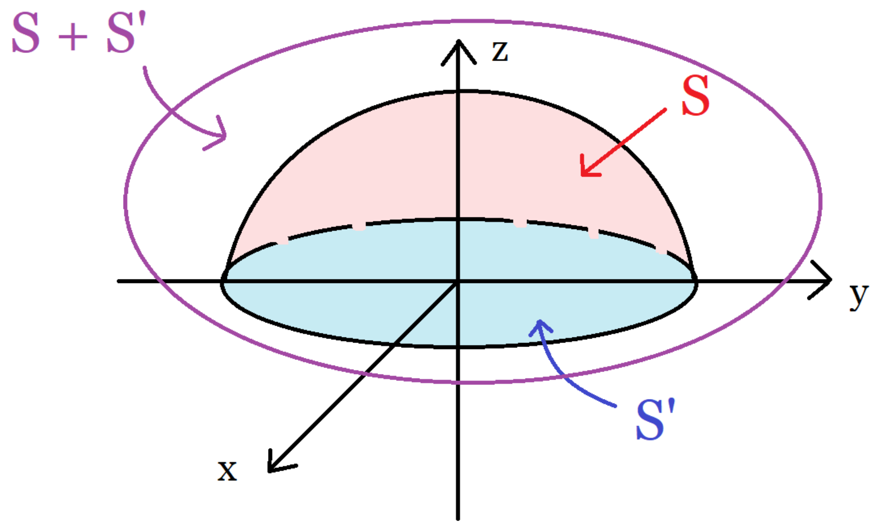
Example 3: (Tricky!)

$$\int\int_S F \cdot d\mathbf{S}$$

$$F = \left\langle z^2 x, \frac{1}{3}y^3 + \tan(z), x^2 z + y^2 \right\rangle$$

S : Top half of sphere $x^2 + y^2 + z^2 = 1$

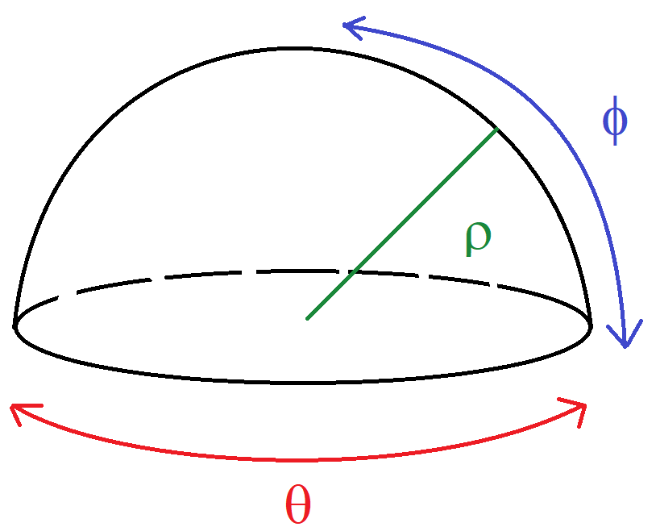
(STEP 1) **Picture:**



WARNING: S is not closed! (doesn't include the bottom lid), so need to close it!

Let $S' =$ bottom disk, then $S + S'$ (= Top Sphere + Bottom Disk) is closed, so by the Divergence Theorem:

$$
\begin{aligned}
\int\int_{S+S'} F\cdot d\mathbf{S} &= \int\int\int_E \operatorname{div}(F)dxdydz \\
&= \int\int\int_E \left(z^2x\right)_x + \left(\frac{1}{3}y^3 + \tan(z)\right)_y + \left(x^2z + y^2\right)_z \, dxdydz \\
&= \int\int\int_E z^2 + y^2 + x^2 \, dxdydz \\
&= \int_0^{\frac{\pi}{2}}\int_0^{2\pi}\int_0^1 \rho^2\,\rho^2\sin(\phi)\,d\rho d\theta d\phi \\
&= 2\pi\left(\int_0^{\frac{\pi}{2}}\sin(\phi)d\phi\right)\left(\int_0^1 \rho^4 d\rho\right) \\
&= \frac{2\pi}{5}
\end{aligned}
$$



(STEP 2)

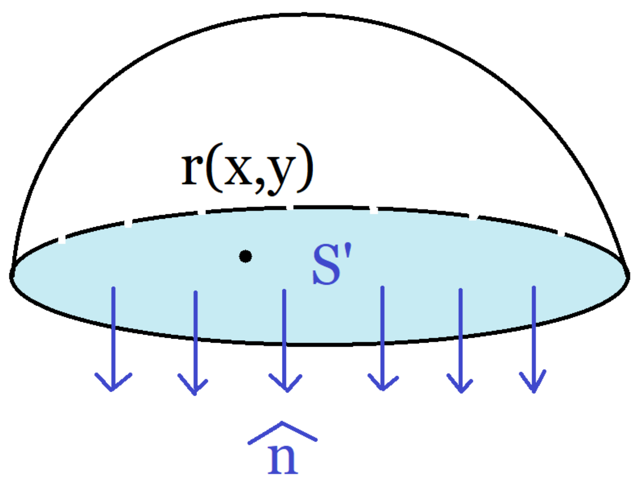
$$\int\int_{S+S'} F \cdot d\mathbf{S} = \frac{2\pi}{5}$$

$$\underbrace{\int\int_{S} F \cdot d\mathbf{S}}_{WTF} + \int\int_{S'} F \cdot d\mathbf{S} = \frac{2\pi}{5}$$

$$\int\int_{S} F \cdot d\mathbf{S} = \frac{2\pi}{5} - \int\int_{S'} F \cdot d\mathbf{S}$$

(STEP 3) $\int\int_{S'} F \cdot d\mathbf{S}$

WARNING: Outward orientation takes precedence over upward orientation here!



So make sure that $\hat{n}$ points **downwards** here!

(1) **Parametrize**

$r(x,y) = \langle x, y, 0\rangle$ (or use polar coordinates, that's ok too)

(2) **Slopes**

$$\begin{aligned} r_x &= \langle 1, 0, 0\rangle \\ r_y &= \langle 0, 1, 0\rangle \end{aligned}$$

(3) **Normal Vector**

$$\begin{aligned} \hat{n} =& r_x \times r_y \\ =& \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} \\ =& \langle 0, 0, 1\rangle \end{aligned}$$

Since we want $\hat{n}$ to point downwards, we choose $\hat{n} = \langle 0, 0, -1\rangle$

(4)

$$\begin{aligned} \int\int_{S'} F \cdot d\mathbf{S} &= \int\int_D F \cdot \hat{n} dx dy \\ &= \int\int_D \left\langle 0x^2 + \frac{1}{3}y^3 + \tan(0), x^2(0) + y^2 \right\rangle \cdot \langle 0, 0, -1\rangle \, dx dy \\ &= \int\int_D -y^2 dx dy \qquad D: \text{ Disk of radius 1} \\ &= \int\int_D -r^2 \sin^2(\theta) r dr d\theta \\ &= \cdots \\ &= -\frac{\pi}{4} \end{aligned}$$

(STEP 4) **Answer:** From STEP 2, we have:

$$\begin{aligned}
\int\int_S F \cdot d\mathbf{S} &= \underbrace{\frac{2\pi}{5}}_{\text{STEP 1}} - \int\int_{S'} F \cdot d\mathbf{S} \\
&= \frac{2\pi}{5} - \underbrace{\left(-\frac{\pi}{4}\right)}_{\text{STEP 3}} \\
&= \frac{13\pi}{20}
\end{aligned}$$