LECTURE 22: THE DIVERGENCE THEOREM (II)

Let’s quickly recap what we know about surface integrals (I know those topics get confusing very quickly)

1. Recap about Surface Integrals
   (1) Every surface $S$ has a normal vector

   Normal Vector
   \[
   \hat{n} = r_u \times r_v
   \]

   \[
   \hat{n} = r_u \times r_v
   \]

   (2) Surface Integral of a Vector Field
Surface Integral of a Vector Field

\[ \iint_{S} F \cdot dS = \iint_{D} F \cdot \hat{n} dudv = \iint_{D} F \cdot (r_{u} \times r_{v}) dudv \]

(Again, the idea is that you sum up the values of \( F \) on the surface \( S \), by dotting \( F \) with the normal vector \( \hat{n} \))

(3) Mini-Parallelograms

\[ dS = \|r_{u} \times r_{v}\| \ dudv = \|\hat{n}\| \ dudv \]
(4) Surface Integral of a function

Surface Integral of $f$

\[
\int \int \int_S f \, dS = \int \int_D f \left\| r_u \times r_v \right\| \, du \, dv = \int \int_D f \left\| \hat{n} \right\| \, du \, dv
\]

(Summing up a function $f$ over a surface)

2. The Adult Surface Integral

Question: Are $\int \int_S F \cdot dS$ and $\int \int_S f \, dS$ related? Yes, and this will be very useful for today.

Definition

\[
n = \frac{\hat{n}}{\left\| \hat{n} \right\|} = \text{Unit normal vector (Length = 1)}
\]
Now let’s look again at our surface integral:

\[
\int \int_S F \cdot dS = \int \int_D F \cdot \hat{n} \, dudv
\]

\[
= \int \int_D F \cdot \underbrace{\hat{n}}_{\parallel \hat{n} \parallel} \underbrace{\parallel \hat{n} \parallel}_{dS} \, dudv
\]

\[
= \int \int_S F \cdot n \, dS
\]

**Adult Surface Integral**

\[
\int \int_S F \cdot dS = \int \int_S F \cdot n \, dS
\]
So the surface integral of the vector field $F$ is the surface integral of the function $F \cdot n$.

And again, this really expresses the fact that we’re summing up the values of $F$ over the surface $S$.

3. An Important Normal Vector

Because of this formula, it’s important to find $n$ for some surfaces. Luckily there’s one surface where $n$ is easy to find.

**Example 1:**

Find $n$ where $S$ is the sphere $x^2 + y^2 + z^2 = r^2$
Notice \( \hat{n} = \langle x, y, z \rangle \)

(technically, it should be \( c \langle x, y, z \rangle \), since \( \hat{n} \) is proportional to \( \langle x, y, z \rangle \), but it gives you the same result)

Therefore:

\[
n = \frac{\langle x, y, z \rangle}{\|\langle x, y, z \rangle\|} = \frac{\langle x, y, z \rangle}{\sqrt{x^2 + y^2 + z^2}} = \frac{1}{r} \langle x, y, z \rangle
\]

**Fact:**

For the sphere of radius \( r \):

\[
n = \frac{1}{r} \langle x, y, z \rangle
\]

4. **Volumes**
Let’s use this fact and the divergence theorem to get an OMG example! (Very similar to the section on Areas when we did Green’s theorem)

### Recall: Divergence Theorem

\[ \int \int_S F \cdot dS = \int \int \int_E \text{div}(F) \, dx \, dy \, dz \]

Now **IF** \( F \) is chosen such that \( \text{div}(F) = 1 \), then

\[ \int \int_S F \cdot dS = \int \int \int_E 1 = \text{Vol}(E) \]

Many choices for \( F \): \( F = \langle x, 0, 0 \rangle, \langle 0, y, 0 \rangle, \langle 0, 0, z \rangle \) and others

**Best choice**: (most balanced): \( F = \frac{1}{3} \langle x, y, z \rangle \), which gives us:

### Fact: (Memorize)

\[ \text{Vol}(E) = \int \int_S F \cdot dS \quad F = \frac{1}{3} \langle x, y, z \rangle \]

Let’s use this to find the Volume of a ball!

### OMG Example 2:

Find the Volume of a Ball of Radius \( r \)
\[ \text{Vol}(E) = \int \int_S F \cdot dS \quad F = \frac{1}{3} \langle x, y, z \rangle \]

\[ = \int \int_S F \cdot n \, dS \quad \text{Adult Surface Integral} \]

\[ = \int \int_S \frac{1}{3} \langle x, y, z \rangle \cdot \frac{1}{r} \langle x, y, z \rangle \, dS \quad \text{S is a sphere of radius } r \]

\[ = \left( \frac{1}{3} \right) \left( \frac{1}{r} \right) \int \int_S x^2 + y^2 + z^2 \, dS \]

\[ = \frac{1}{3r} \int \int_S r^2 \, dS \]

\[ = \frac{r^2}{3r} \int \int_S 1 \, dS \]

\[ = \frac{r}{3} \text{Area}(S) \]

\[ = \frac{4 \pi r^2}{3} \]

\[ = \frac{4}{3} \pi r^3 \quad \text{EFFORTLESS!} \]
(On the HW, I ask you to derive the surface area of a sphere in a similar way)

**OMG Remark:**
Notice that \((\frac{4}{3}\pi r^3)' = 4\pi r^2\). Is this a coincidence? Actually no! (see HW or this video; this result is true in any dimensions)

5. **The Genesis of Laplace**
If you combine the divergence with the gradient \(\nabla\), then you get a monster called the Laplacian:

\[
\text{div}(\nabla f) = \text{div}(\langle f_x, f_y, f_z \rangle) = (f_x)_x + (f_y)_y + (f_z)_z = f_{xx} + f_{yy} + f_{zz} = \Delta f
\]

**Fact:**
\[
\text{div}(\nabla f) = \Delta f = f_{xx} + f_{yy} + f_{zz}
\]

And associated to this is:

**Laplace’s Equation:**
\[
\Delta f = 0
\]

The next example explains (sort of) where Laplace’s equation comes from:
Example 3:

Suppose $\Delta f = 0$ in $E$ and define $F = \nabla f$. Show that $\int \int_S F \cdot dS = 0$

\[
\int \int_S F \cdot dS = \int \int \int_E \text{div}(F) dxdydz
\]
\[
= \int \int \int_E \text{div}(\nabla f) dxdydz \quad (F = \nabla f)
\]
\[
= \int \int \int_E \Delta f dxdydz \quad (\text{Definition})
\]
\[
= \int \int \int_E 0 dxdydz
\]
\[
=0
\]

**Interpretation:** If $f \Delta f = 0$, then $\int \int_S F \cdot dS = 0$ means that $F = \nabla f$ is in equilibrium (net flux = 0, Flow in = Flow out)
Note: $\Delta f = 0$ is the single, most important equation in the universe!
Here are some applications:

(1) $\Delta f = 0$ measures a fluid in equilibrium

(2) The solution $f(x, y, z)$ of $\Delta f = 0$ gives you the temperature of a metal solid $E$ after a long time (think of a metal plate that you took out of the oven and let it sit for a long time)

(3) It’s because of Laplace’s equation that I got my PhD. If you’re curious about what my thesis was about, check out The PDE that gave me the PhD.

(4) Really cool application: Suppose you start at a point $(x, y, z)$ and you perform Brownian motion (= drunken motion) until you hit a wall at $(x^*, y^*, z^*)$, where you pay a penalty $g(x^*, y^*, z^*)$. 
Analogy: You’re driving drunk (please don’t do this!), and $g(x^*, y^*, z^*)$ is the money that you have to pay to the insurance.

This is a random process, but we can still calculate its average value of this event.
Let \( f(x, y, z) = \) Average payoff/penalty you get, starting at \((x, y, z)\).

**Cool Fact**

Then \( f \) solves \( \Delta f = 0 \)

**Some related equations**

Here \( f = f(x, y, z, t) \) ((\(x, y, z\)) is position and \( t \) is time)

1. \( f_t = \Delta f \) (Heat equation; Temperature of metal plate for all time)
2. \( f_{tt} = \Delta f \) (Wave equation; Height of a wave at \((x, y, z)\) and time \( t \))

**VERY** different equations! One \( t \) makes a big difference!