LECTURE 22: THE DIVERGENCE THEOREM (II)

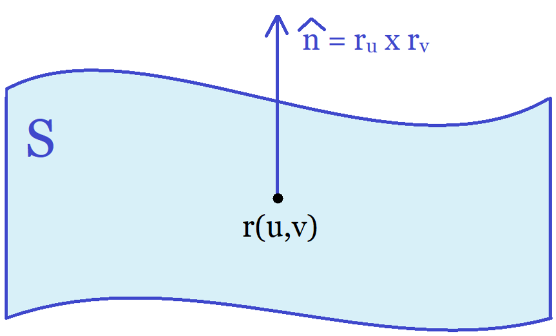
Let's quickly recap what we know about surface integrals (I know those topics get confusing very quickly)

1. Recap about Surface Integrals

(1) Every surface S has a normal vector

Normal Vector

$$\hat{n} = r_u \times r_v$$



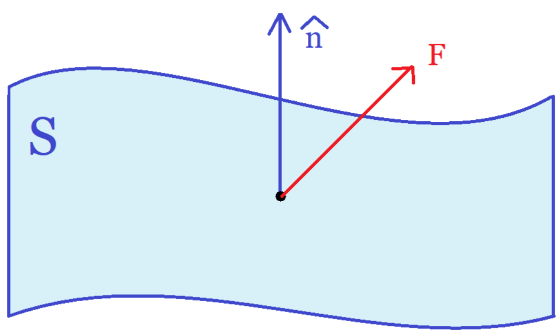
(2) **Surface Integral of a Vector Field**

Date: Monday, March 2, 2020.

Surface Integral of a Vector Field

$$\int\int_S F \cdot d\mathbf{S} = \int\int_D F \cdot \hat{n} dudv = \int\int_D F \cdot (r_u \times r_v) dudv$$

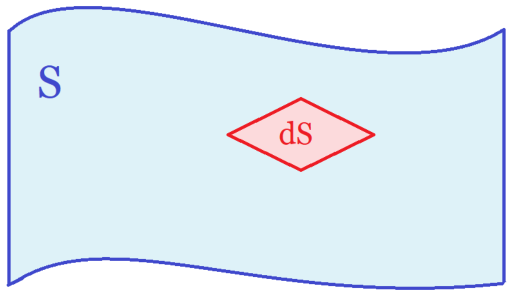
(Again, the idea is that you sum up the values of F on the surface S, by dotting F with the normal vector $\hat{n}$)



(3) **Mini-Parallelograms**

Mini-Parallelograms

$$dS = \|r_u \times r_v\| \, dudv = \|\hat{n}\| \, dudv$$



(4) **Surface Integral of a function**

Surface Integral of f

$$\int\int_S f\,dS = \int\int_D f\,\underbrace{\|r_u \times r_v\|\,dudv}_{dS} = \int\int_D f\,\|\hat{n}\|\,dudv$$

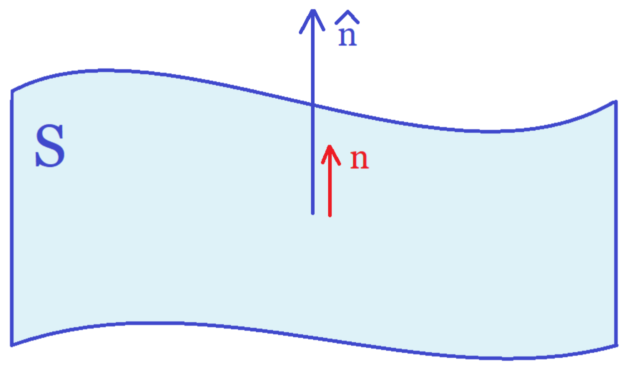
(Summing up a function f over a surface)

2. The Adult Surface Integral

Question: Are $\int\int_S F\cdot d\mathbf{S}$ and $\int\int_S f dS$ related? Yes, and this will be very useful for today.

Definition

$$n = \frac{\hat{n}}{\|\hat{n}\|} = \textbf{ Unit} \text{ normal vector (Length} = 1)$$

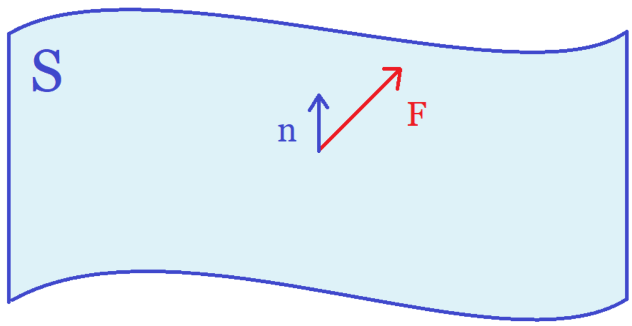


Now let's look again at our surface integral:

$$\begin{aligned}\int\int_S F \cdot d\mathbf{S} &= \int\int_D F \cdot \hat{n}\,dudv \\ &= \int\int_D F \cdot \underbrace{\frac{\hat{n}}{\|\hat{n}\|}}_{n} \underbrace{\|\hat{n}\|\,dudv}_{dS} \\ &= \int\int_S F \cdot n \; dS\end{aligned}$$

Adult Surface Integral

$$\int\int_S F \cdot d\mathbf{S} = \int\int_S F \cdot n\, dS$$



So the surface integral of the **vector field** F is the surface integral of the **function** $F \cdot n$.

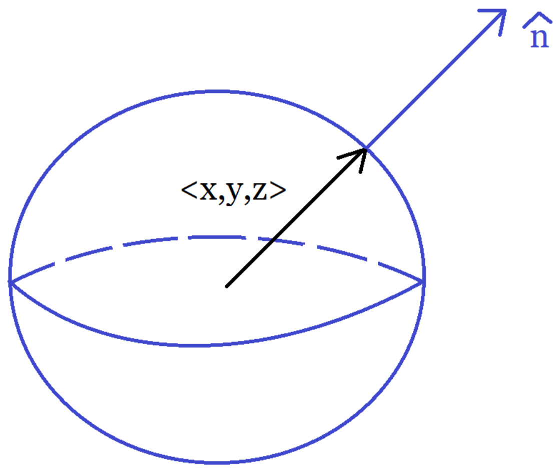
And again, this really expresses the fact that we're summing up the values of F over the surface S.

3. An Important Normal Vector

Because of this formula, it's important to find n for some surfaces. Luckily there's one surface where n is easy to find.

Example 1:

Find n where S is the sphere $x^2 + y^2 + z^2 = r^2$



Notice $\hat{n} = \langle x, y, z \rangle$

(technically, it should be $c \langle x, y, z \rangle$, since $\hat{n}$ is proportional to $\langle x, y, z \rangle$, but it gives you the same result)

Therefore:

$$n = \frac{\langle x, y, z \rangle}{\| \langle x, y, z \rangle \|} = \frac{\langle x, y, z \rangle}{\sqrt{x^2 + y^2 + z^2}} = \frac{1}{r} \langle x, y, z \rangle$$

Fact:

For the sphere of radius r:

$$n = \frac{1}{r} \langle x, y, z \rangle$$

4. Volumes

Let's use this fact and the divergence theorem to get an OMG example! (Very similar to the section on Areas when we did Green's theorem)

Recall: Divergence Theorem

$$\int\int_S F \cdot d\mathbf{S} = \int\int\int_E \operatorname{div}(F) dxdydz$$

Now **IF** F is chosen such that $\operatorname{div}(F) = 1$, then

$$\int\int_S F \cdot d\mathbf{S} = \int\int\int_E 1 = Vol(E)$$

Many choices for F**:** $F = \langle x, 0, 0 \rangle, \langle 0, y, 0 \rangle, \langle 0, 0, z \rangle$ and others

Best choice: (most balanced): $F = \frac{1}{3} \langle x, y, z \rangle$, which gives us:

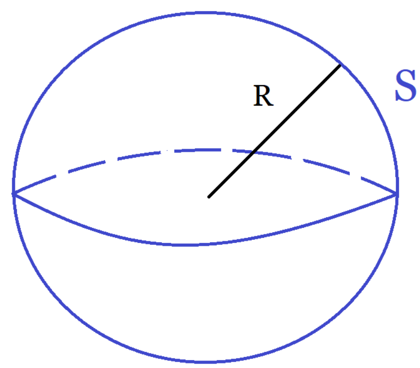
Fact: (Memorize)

$$Vol(E) = \int\int_S F \cdot d\mathbf{S} \qquad F = \frac{1}{3} \langle x, y, z \rangle$$

Let's use this to find the Volume of a ball!

OMG Example 2:

Find the Volume of a Ball of Radius r



$$
\begin{aligned}
Vol(E) &= \int\int_S F \cdot d\mathbf{S} \quad F = \frac{1}{3}\langle x, y, z\rangle \\
&= \int\int_S F \cdot n\, dS \quad \text{Adult Surface Integral} \\
&= \int\int_S \underbrace{\frac{1}{3}\langle x, y, z\rangle}_{F} \cdot \underbrace{\frac{1}{r}\langle x, y, z\rangle}_{n} dS \quad S \text{ is a sphere of radius } r \\
&= \left(\frac{1}{3}\right)\left(\frac{1}{r}\right)\int\int_S x^2 + y^2 + z^2 dS \\
&= \frac{1}{3r}\int\int_S r^2 dS \\
&= \frac{r^2}{3r}\int\int_S 1 dS \\
&= \frac{r}{3} Area(S) \\
&= \frac{r}{3} 4\pi r^2 \\
&= \frac{4}{3}\pi r^3 \qquad \color{red}{EFFORTLESS!}
\end{aligned}
$$

(On the HW, I ask you to derive the surface area of a sphere in a similar way)

OMG Remark:

Notice that $\left(\frac{4}{3}\pi r^3\right)' = 4\pi r^2$. Is this a coincidence? Actually no! (see HW or this video; this result is true in any dimensions)

5. The Genesis of Laplace

If you combine the divergence with the gradient ∇, then you get a monster called the Laplacian:

$$\operatorname{div}(\nabla f) = \operatorname{div}\left(\langle f_x, f_y, f_z\rangle\right) = (f_x)_x + (f_y)_y + (f_z)_z = f_{xx} + f_{yy} + f_{zz} = \Delta f$$

Fact:

$$\operatorname{div}(\nabla f) = \Delta f = f_{xx} + f_{yy} + f_{zz}$$

And associated to this is:

Laplace's Equation:

$$\Delta f = 0$$

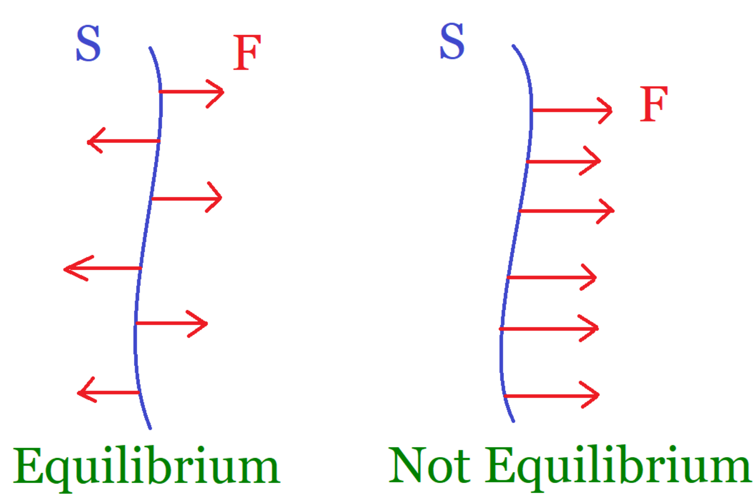
The next example explains (sort of) where Laplace's equation comes from:

Example 3:

Suppose $\Delta f = 0$ in E and define $F = \nabla f$. Show that $\int\int_S F \cdot d\mathbf{S} = 0$

$$
\begin{aligned}
\int\int_S F \cdot d\mathbf{S} &= \int\int\int_E \operatorname{div}(F) dxdydz \\
&= \int\int\int_E \operatorname{div}(\nabla f) dxdydz \qquad (F = \nabla f) \\
&= \int\int\int_E \Delta f dxdydz \qquad \text{(Definition)} \\
&= \int\int\int_E 0 dxdydz \\
&= 0
\end{aligned}
$$

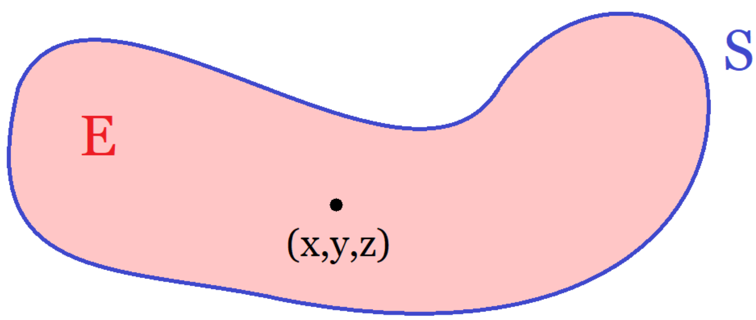
Interpretation: If f $\Delta f = 0$, then $\int\int_S F \cdot d\mathbf{S} = 0$ means that $F = \nabla f$ is in equilibrium (net flux = 0, Flow in = Flow out)



Note: $\Delta f = 0$ is the single, most important equation in the universe! Here are some applications:

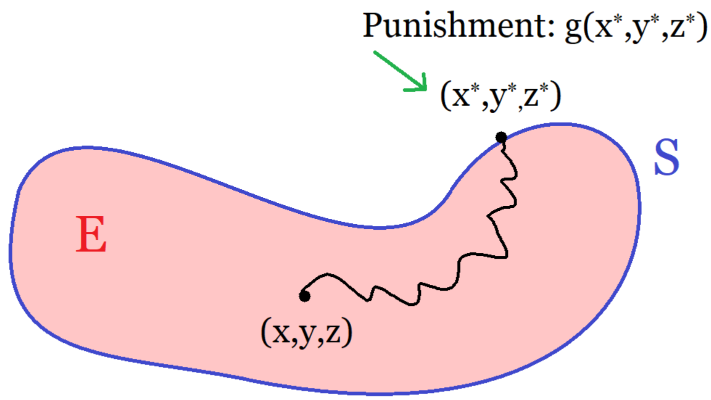
(1) $\Delta f = 0$ measures a fluid in equilibrium

(2) The solution $f(x, y, z)$ of $\Delta f = 0$ gives you the temperature of a metal solid E after a long time (think of a metal plate that you took out of the oven and let it sit for a long time)



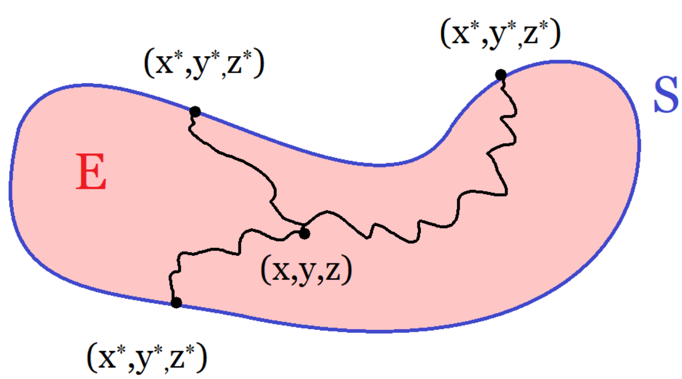
(3) It's *because* of Laplace's equation that I got my PhD. If you're curious about what my thesis was about, check out The PDE that gave me the PhD.

(4) **Really cool application:** Suppose you start at a point (x, y, z) and you perform Brownian motion (= drunken motion) until you hit a wall at $(x^\star, y^\star, z^\star)$, where you pay a penalty $g(x^\star, y^\star, z^\star)$.



(Analogy: You're driving drunk (please don't do this!), and $g(x^{\star}, y^{\star}, z^{\star})$ is the money that you have to pay to the insurance)

This is a random process, but we can still calculate iyts average value of this event.



Let $f(x, y, z)$ = Average payoff/penalty you get, starting at (x, y, z).

Cool Fact

Then f solves $\Delta f = 0$

Some related equations

Here $f = f(x, y, z, t)$ ((x, y, z) is position and t is time)

(1) $f_t = \Delta f$ (Heat equation; Temperature of metal plate for *all* time)

(2) $f_{tt} = \Delta f$ (Wave equation; Height of a wave at (x, y, z) and time t)

VERY different equations! One t makes a big difference!