FINAL EXAM – STUDY GUIDE

The Final Exam takes place at the following times:

(1) LEC A (MWF 11-12 in 104 RH): Friday, March 20, 8-10 AM

(2) LEC F (MWF 10-11 in 1600 DBH): Monday, March 16, 10:30 AM - 12:30 PM

It will be held remotely, through Canvas. Instructions on how to take the remote final will be provided on a separate document. **No books/notes/calculators/cheat sheets/collaboration is allowed.** The final counts for 50 % of your grade and can replace your midterm grade if you do better on it. It covers sections $15.2 - 15.3$, $15.6 - 15.9$, and all of Chapter 16, with heavy emphasis on Chapter 16. This is the study guide for the exam, and is just meant to be a *guide* to help you study, just so we're on the same place in terms of expectations. For a more thorough study experience, look at all the lectures, the mock exam, the final from Fall 2018, and the suggested homework.

The exam will *tentatively* have 8-9 problems in total. Most of them should be doable if you know the material, maybe the last one will be trickier than the rest. **80-90 % of the problems will be from Chapter 16, so make sure to study that chapter extra thoroughly**. That said, don't forget about Chapter 15, because the techniques in Chapter 15 (polar coordinates, spherical coordinates, etc.) might appear as Chapter 16-type questions! And as before, know the 6 surfaces on Table 1 of section 12.6; I could ask you about any of those surfaces and you might be forced to draw a picture.

YouTube playlists: Check out the following YouTube playlists from the Chapter 15 and 16 material:

- Chapter 15: Multiple Integrals
- Chapter 16: Vector Calculus
- Vector Calculus Overview

Note: I *will* give you the following equations for spherical coordinates, so you do **NOT** have to memorize them:

$$\begin{aligned} x =& \rho \sin(\phi)\cos(\theta) \\ y =& \rho \sin(\phi)\sin(\theta) \\ z =& \rho\cos(\phi) \\ \text{Jac} =& \rho^2 \sin(\phi) \end{aligned}$$

Useful trig identities to know:

(1) $\sin^2(x) + \cos^2(x) = 1$

(2) $1 + \tan^2(x) = \sec^2(x)$

(3) $\cos(-x) = \cos(x), \sin(-x) = -\sin(x)$

(4) $\sin(2x) = 2\sin(x)\cos(x)$, $\cos(2x) = \cos^2(x) - \sin^2(x)$

Useful integration techniques to know:

(1) $u-$substitution

(2) Integration by parts, but I'll only ask you about easy cases like $\int xe^x$

(3) Trig integrals like $\int \cos^3(x)\sin(x)dx$

(4) $\int \cos^2(x)dx$, $\int \sin^2(x)dx$; check out this video in case you forgot

(5) $\int \sqrt{x^2+1}$ (using trig substitution,; the only situation where you might use it is for surface areas); check out this video in case you forgot

(6) **No** partial fractions

Sections 15.2, 15.3, 15.6, 15.7: Double and Triple Integrals

- Just quickly brush up on those sections, most of it should be familiar to you. I probably won't ask questions per se from those sections, but they might appear as part of other questions (for example, if you're using the divergence theorem, you'll need to know how to calculate triple integrals), so you're still responsible for knowing them.

- Find the double integral of a function over a general region. Sometimes the region is a vertical region, so you'll have to do Smaller $\leq y \leq$ Bigger, and sometimes it's a horizontal region, in which case you have to solve for x in terms of y and do Left $\leq x \leq$ Right (15.2.18, 15.2.19)

- Evaluate an integral by changing the order of integration (15.2.52, 15.2.56, Order of Integration)

- Find the volume of a given solid (15.2.25, 15.2.27)

- Find the average value of a function (15.2.62)

- Evaluate an integral by changing to polar coordinates. Remember that this is excellent if you see x^2+y^2 or your region is a disk or a wedge or a ring (annulus). **Don't forget about the**

r**!!!** (15.3.9, 15.3.10, 15.3.13, 15.3.19-26 Polar Integral, Integral over a ring)

- Calculate $\int_{-\infty}^{\infty} e^{-x^2} dx$ (you can check out this YouTube video). Don't worry about the problem with $\int_{-\infty}^{\infty} \sin(x^2) dx$
- Find a triple integral over a general $3D$ region; good examples of regions are either tetrahedrons, or regions between two surfaces (like in lecture); *usually* you have to do Smaller $\leq z \leq$ Bigger
- Remember that sometimes you region faces the $y-$ direction (the book calls this type 3), in which case you have to do Left $\leq y \leq$ Right or the $x-$direction (type 2), in which case you have to do Back $\leq x \leq$ Front.
- Review the problems from Lecture 3, those are very good sample questions, as well as problems 13, 15, 17, 18, 19-22 in section 15.6
- Find the average value of a function (15.6.53, 15.6.54)
- Remember the problem about the problem of intersection of two cylinders: Intersection of 2 Cylinders
- Calculate triple integrals using cylindrical coordinates; problems 17 through 24 in 15.7 are excellent practice problems. Cylindrical coordinates are basically polar coordinates, but with an extra factor of z, also see Cylindrical Coordinates
- Remember that cylindrical coordinates are useful if you have cylinders or if you see $\sqrt{x^2 + y^2}$

Section 15.8: Triple Integrals in Spherical Coordinates

- Again, I will provide you with the equations for spherical coordinates, so no need to memorize them, but:

- You do **NOT** need to know how to derive the equations for spherical coordinates any more!
- Sketch surfaces with given spherical coordinates, like $\rho = 3$ or $\phi = \frac{\pi}{6}$
- Calculate integrals and volumes using spherical coordinates; problems 21, $22 - 28$, and 30 are excellent practice problems. The problems in lecture are also great, and you can check out Volume of an Ice Cream Cone and Spherical Coordinates Example
- Remember that spherical coordinates are great for spheres and anything that involves $\sqrt{x^2 + y^2 + z^2}$.
- I could ask you to find the mass of the sun again :) Mass of the Sun

Section 15.9: Change of Variables in Multiple Integrals

- There are two kinds of change of variables I could ask you:
- One where you choose u and v, like in $\int\int \cos\left(\frac{y-x}{y+x}\right) dxdy$. In that case, I will **NOT** give you u and v and you'll have to figure it out, either by using the function, or by using the region (like in the problem on the mock midterm). Problems $23 - 28$ are great sample problems, as well as the problems in lecture, the problem on the mock midterms/final and the actual midterm and the final from 2018.
- Or one where x and y are in terms of u and v, like Problems 15 $-$ 19. In that case, I will **give** you x and y.
- Don't forget about the absolute value!!!

- The easiest way to memorize the Jacobian definition is by using $dx = \frac{dx}{du}du$, or $du = \frac{du}{dx}dx$. Remember the motivating examples in lecture!
- Know how to rederive the Jacobians for polar, cylindrical, and spherical coordinates (see Lecture 8 or the Additional Problems in HW 3)
- You might also want to check out my YouTube videos: Jacobian 1, Jacobian 2, and Change of Variables
- I could ask you about Hyperbolic coordinates (see last problem on the midterm of 2018), but I would give you all the formulas you'd need to know for it. Check out: Hyperbolic Coordinates

Section 16.1: Vector Fields

- I won't ask you anything about this section since it's just an introduction, so you can skip it if you want
- Here's a great overview of all the topics in chapter 16: Vector Calculus Overview

Section 16.2: Line Integrals

- Remember the three important parametrizations in Lecture 10: The circle, the line segment, and the function, see Parametric Equations
- Calculate $\int_C f(x,y)ds$ or $\int_C f(x,y,z)ds$, like problems 3, 4, 9 – 12. And remember that it represents the area under a fence. Check out the following videos for examples: Example 1, Example 2, Example 3, Example 4
- The easiest way to memorize this is by using $ds = \sqrt{(dx)^2 + (dy)^2}$ (in the two-dimensional case), see Line Integral Derivation

- Calculate $\int_C Pdx + Qdy$ or $\int_C Pdx + Qdy + Rdz$, like problems 5, 6, 8, 13 -16. Check out this video for an example.
- The easiest way to memorize this is by using $dx = \frac{dx}{dt}dt = x'(t)dt$
- You don't need to know the interpretation given in Lecture 11 with the shadows
- Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$, like problems 19-22. Check out this video for an example.
- Find the work done by a vector field $\mathbf{F}$ on a curve C, like $40 - 41$
- Also check out 16.2.50

Section 16.3: The Fundamental Theorem for Line Integrals

- Know that in $2D$, $\mathbf{F} = \langle P, Q \rangle$ is conservative if and only if $P_y = Q_x$. (Peyam is Quixotic), like in problems 3-10
- (16.5) Know that in $3D$, $\mathbf{F} = \langle P, Q, R \rangle$ is conservative if and only if $\operatorname{curl} \mathbf{F} = \langle 0, 0, 0 \rangle$.
- Find a function f such that $\mathbf{F} = \nabla f$, like in $12 - 18$. This works in any dimensions. You can use the method in lecture or in the book, whichever you prefer.
- Use the FTC for line integrals to calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$, like in $12 - 18$. Check out this video for an example. Both the 2D case and the 3D case are fair game for the exam.
- Know the neat fact that $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path if and only if $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed curve C if and only if $\mathbf{F}$

is conservative. You don't need to know their proofs. See also problems 19 and 20.

- If there is a hole, I will explicitly mention that there is one. Check out 35 for an interesting example

- The problems in Lecture 13 are good problems as well. See for instance the following videos: FTC Example, FTC 3D Example

Section 16.4: Green's Theorem

- Use Green's theorem to calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$. Remember that this only works for closed curves C. See the problems in Lecture 15, as well as Problems 1-14. Check out this video for an example. Don't worry about the orientation of the curve; for Green's theorem, I won't trick you with that.

- Use Green's theorem to find the area enclosed by a curve. The easiest way to remember this is:

$$A = \frac{1}{2} \int_C \begin{vmatrix} x & y \\ dx & dy \end{vmatrix} = \frac{1}{2} \int_C x dy - y dx$$

  You don't need to know the other formulas! For practice, look at Example 3, Problem 21, and the AP in Homework 6, and this video: Area of Ellipse

- You don't need to memorize the formula for the area of a polygon (problem 21), but you need to know how to derive it, see this video: Area of Polygon

- Also look at Problem 27 and the example in lecture with holes. The point of this problem is that since we get the same answer for any closed curve C, choose the easiest one, usually a circle. I will explicitly mention that there is a hole in this problem.

- You don't need to know about winding numbers

Section 16.6: Parametric Surfaces and their Areas

- Know how to parametrize familiar surfaces, like the cylinder, spheres, planes, and functions. See the examples in Lecture 17, as well as problems 20, 23, 24, 26
- Ignore the example with solids of revolution
- Sometimes the parametrization will be given (like the helicoid), but sometimes you have to figure it out on your own.
- Find the equation of the tangent plane to a surface, see problems $33 - 36$. Check out this video for an example.
- Know that $\hat{n} = r_u \times r_v$ represents the normal vector to your surface, and $dS = \|r_u \times r_v\|\, dudv$ represents the area of a tiny parallelogram.
- Find surface areas of parametric curves; the examples in Lecture 18 are good examples. Problems $39 - 48$ are all good practice questions. Check out this video for an example.
- The easiest way to remember $\int\int_D \|r_u \times r_v\|\, dudv$ is by writing the area as $\int\int_S 1 dS$ and using the definition of dS.
- **DON'T** memorize the formula for the surface area of the graph of a function (Formula 9 in section 16.6), it's *much* easier to derive by using the parametrization $\mathbf{r}(x, y) = \langle x, y, f(x, y)\rangle$
- You can ignore 64 (the problem with a donut), but it's a good extra challenge.
- In HW7, you can ignore AP1, but try AP2.

Section 16.7: Surface Integrals

- Find surface integrals of functions. The easiest way to remember the formula is by starting with $\int\int_S f dS$ and by using the formula $dS = \|r_u \times r_v\| \, dudv$. Again, think height times base, where your height is f and your base is dS (the mini-parallelogram)

- Problems $5 - 20$ and the problems in Lecture 19 are all good practice questions. Check out this video for an example

- Again, **DON'T** memorize formula 4 in section 16.7, it's much easier just to use the parametrization $\mathbf{r}(x, y) = \langle x, y, f(x, y)\rangle$ and to use the definition of the surface integral

- Find surface integrals of vector fields. The easiest way to memorize it is by writing it as $\int\int_D \mathbf{F} \cdot \hat{n} dudv$ and by using $\hat{n} = \mathbf{r}_u \times \mathbf{r}_v$.

- In this case, you're dotting $\mathbf{F}$ with the normal vector, whereas for line integrals you're dotting $\mathbf{F}$ with the direction vector! That's because the direction vector is to a line what the normal vector is to a curve

- The problems in Lecture 20 are good practice problems, as well as Problems $21 - 32$. Check out this video for an example

- **PLEASE DON'T** memorize Formula 10 in section 16.7, I'm begging you!!! Again, it's so much easier to use $\mathbf{r}(x, y) = \langle x, y, g(x, y)\rangle$

- **For surface integrals, orientation matters!!!! Unless otherwise specified, make sure that your normal vector points upwards (for surfaces) or outwards (for closed surfaces like the sphere)**. Most of the time this just amounts to checking that the third component of $\hat{\mathbf{n}}$ is ≥ 0, but a picture should really help.

- Know the "adult" definition of the surface integral $\int\int_S \mathbf{F} \cdot d\mathbf{S} = \int\int \mathbf{F} \cdot \mathbf{n} dS$, where $\mathbf{n} = \frac{\hat{\mathbf{n}}}{\|\hat{\mathbf{n}}\|}$ is the **unit** normal vector to S. You don't need to know how to rederive this.

- Know how to find $\mathbf{n}$ for the sphere (see Lecture 22).

- I will tell you when to use the adult definition of the surface integral; it's usually only used for theoretical questions, you never use it in computational questions.

Section 16.9: The Divergence Theorem

- (16.5) Find the Divergence of a vector field. Problems 1−8 in 16.5 are good practice. While you're at it, you might also want to check out Problem 25 in 16.5 and AP1 in Homework 7.

- Use the divergence theorem to calculate surface integrals. The problems in Lecture 21, as well as the problems 5−12 in 16.9 are good practice. Check out this video for an example

- **Caution:** The divergence theorem only works for **closed** surfaces. If S is not closed, then make sure to close it! See the last example in Lecture 21, as well as problems 17 and 18 and the problem on the mock final to see how to deal with open surfaces.

- **Remember that for orientation, *outward* orientation takes precedence over *upward* orientation.** Again, see the last example in Lecture 21 to see what I mean by that

- Use the divergence theorem to calculate volumes, namely:

$$Vol(E) = \int\int_S \mathbf{F} \cdot d\mathbf{S}$$

where $\mathbf{F} = \frac{1}{3}\langle x, y, z\rangle$. Great practice problems are the problem in Lecture 22, as well as AP3 and AP4 on Homework 9, as well as this video: Derivative of Volume.

- Know that $\text{div}(\nabla f) = \Delta f$ (The Laplacian of f). Check out Example 3 in Lecture 22, as well as AP2 in Homework 9.
- You don't need to know the applications of Laplace's equation given at the end of Lecture 22
- You may also want to check out Problems $25 - 30$ in section 16.9
- Ignore Problem 9(b) on the Fall 2018 final, no one got it right!

Section 16.5: Curl (and Divergence)

- Find the curl of a vector field. Problems $1-8$ are good practice. The easiest way to memorize it is just by remembering that $\text{curl}(\mathbf{F}) = \nabla \times \mathbf{F}$
- Just know that intuitively, the curl represents the rotation of a vector field; you don't need to know the detailed description that I gave in lecture.
- Know that $\mathbf{F}$ is conservative if and only if $\text{curl}(\mathbf{F}) = \langle 0, 0, 0\rangle$. See Problems $13 - 18$. I might combine that with a question about the FTC for line integrals, like in Lecture 23 or the mock final, so at this point it might be good practice to go back to 16.3 and do Problems $15-18$. In such a problem, you **HAVE** to check the curl, it's not enough on the exam just to find f such that $\mathbf{F} = \nabla f$

- Know that $\operatorname{curl}(\nabla f) = \langle 0, 0, 0\rangle$ and $\operatorname{div}(\operatorname{curl}(\mathbf{F})) = 0$; The way to remember this is that if you apply a new topic in the book to the topic right before that, you should get 0. For example, in the book, div comes after curl, that's why $\operatorname{div}(\operatorname{curl}(\mathbf{F})) = 0$.
- Use the preceding fact to show that a vector field cannot be written as a curl, see Problems 19 and 20.
- You might also want to check out Problems $23 - 29$
- Ignore 35 in 16.5

Section 16.8: Stokes' Theorem

- There are two ways to use Stokes' theorem:

(1) Use Stokes to calculate $\int\int_S \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S}$. This just amounts to calculating $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the boundary of S. See Problem 2, the problems in Lecture 24, and the problem on Quiz 10. Problems 3-5 are a bit too complicated, everything will most likely be oriented in the positive $z-$axis.

- **Note: MOST** of the time, C will be oriented counterclockwise, but remember that the best way to check the orientation is that if you walk on the curve C with your head in the direction of the normal vector $\hat{\mathbf{n}}$, then the surface S needs to be on the **LEFT** (wa**L**k left).

- To practice with orientation, check out the last problem in Lecture 24, as well as AP3 in Homework 10, and the problem on the mock final, or see this video: Integral over a barrel
- Honestly, if you remember the orientation of the pictures on pages 2 and 7 of Lecture 24, that should be more than enough; there aren't really any other scenarios that could happen.

- **Note:** Remember that you can actually avoid some of those problems by using the divergence theorem, see the added notes at the end of Lecture 24

(2) Use Stokes to calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$. This just amount to calculating $\int\int_S \text{curl}(\mathbf{F}) \cdot d\mathbf{S}$, where S is (usually) the interior of C. See Problems $7 - 10$ as well as the problems in Lecture 25 for practice.

- Check out this video for an example

Section 16.10: Summary

- Check out the summary in section 16.10
- One of the hard things to know is when to use which FTC. This is the point of the review Lectures $26 - 27$.
- For the donut lecture (Lecture 28), all I want you to understand is how to calculate a surface area, a surface integral, and a volume. The problems I covered in that lecture are too hard to ask on the exam.
- Please also look at the handouts on my website which explain the similarities between the FTC and which give a roadmap of which theorem to use when. Notice that all the FTCs say that the integral of a derivative equals to the function itself, or the integral on the boundary of that function.