LECTURE 1: LAPLACE'S EQUATION

Readings:

- All of Chapter 1 (Introduction)
- Pages 20-21 (Derivation of Laplace's Equation)
- Section 4 of the lecture notes here on Applications of Laplace's Equation
- 2.2.1a Derivation of the Fundamental Solution (pages 21-22)

1. Introduction

Hello everyone and welcome to this reading course about PDEs! My name is Peyam and I'll be your PDE master this quarter!

A couple of words about me: I'm currently a 3rd year Visiting Assistant Professor at UC Irvine (and it's my last quarter here). Before, I went to UC Berkeley, and my PhD advisor was Lawrence C. Evans, who the author of the book we'll be using!

Format of the course: Really chill: Every week there is assigned reading, which you'll do. There won't be any formal meetings due to the coronavirus, but there are plenty of resources which you can use (see below). In some sense, you are designing your own course, you can spend as much time or as little time as you want on it, as long as

Date: Wednesday, March 31, 2020.

you do at least the bare minimum of work also outlined below.

Resources:

- ▶ **Lecture Notes:** The lecture notes (like this one) are just supposed to give you the general idea of the proofs, without giving much detail. Think of them as a supplement - not a substitute - to the book. It's supposed to mimic whatever I would talk about if we had class together.

- ▶ **YouTube Videos:** The YouTube videos on my website cover some of the proofs in more detail. I might make more of them if I have the time, but it's not a guarantee. There's a whole playlist dedicated to this course, check it out: PDE playlist

- ▶ **Suggested Homework:** There is suggested homework (with solutions) if you want more practice

- ▶ **Zoom Office Hours:** On Wednesdays 4-5 PM I have a zoom office hour completely dedicated for you, if you have any questions. Attendance is completely optional. I also have zoom OH on Tuesdays 4-5 PM, but it's mixed with my 140A students. Of course, if you ever want to see me one-on-one, feel free to e-mail me so we can set up a Zoom meeting.

- ▶ **Campuswire:** Like Piazza, but apparently better. It's basically a platform where you can post questions and your classmates can answer them. If you have any questions that you think would benefit everyone, please feel free to post there

- ▶ **E-mail:** Of course, feel free to e-mail me anytime if you have questions

Grading: Very easy, basically show me that you've done the readings. The details are outlined in the syllabus, but you could

- Send me a question about the reading by e-mail
- Write a short paragraph summarizing the reading that you have done for the week, like "This week I learned about the fundamental solution of Laplace's equation, here is the main idea and strategy..."
- Post a question on Campuswire or answer a question that someone else posted
- (as a last resort) E-mail me solutions of two of the problems on the suggested homework

Notes:

(1) Please take this course P/NP!

(2) It's **normal** if the book is hard to read, it really skips a lot of details! On the bright side, we'll go at a really slow pace (except maybe this week).

(3) The devil is really in the details, so don't just skim the book, try to really understand all the details. **It's completely normal to spend 15 mins on one line of the book**

2. What is a PDE?

Reading: Chapter 1

Video: What is a PDE?

A PDE is the multivariable analog of an ODE. In the words of my advisor: *If you solve all PDE, you solve the universe.* What this means is that PDEs are not only incredibly hard to solve, but also incredibly powerful! Each PDE is its own little universe.

Some applications:

(1) **Physics:** Schrödinger's equation in quantum mechanics, Navier-Stokes equations (weather prediction), Maxwell's equations (for light), Einstein's equations (relativity)

(2) **Geometry:** Perelman used PDEs for minimal surfacves to solve the Poincaré conjecture

(3) **Probability:** Brownian Motion, *Stochastic* PDEs, Applications in Finance (Black-Scholes PDE) and Airplane Simulation

(4) **Optimization/Game Theory:** Whenever you're trying to maximize/minimize something, a *Hamilton-Jacobi* equation appears

(5) **Image processing:** The reason your smartphone has such a high image resolution is because of PDEs, also used in MRIs

(6) I literally got a PhD because of a PDE; my PDE is used to model chemical reactions, you can find out more about it here: The PDE that gave me the PhD

In this course, we will study 3 main PDEs:

(1) Laplace's Equation: $\Delta u = 0$

(2) The Heat equation: $u_t = \Delta u$

(3) The Wave equation: $u_{tt} = \Delta u$ (one small difference in t makes a huge difference, like an extra chromosome)

3. Derivation of Laplace's Equation

Reading: Page 20 of section 2.2

Video: Derivation of $\Delta u = 0$

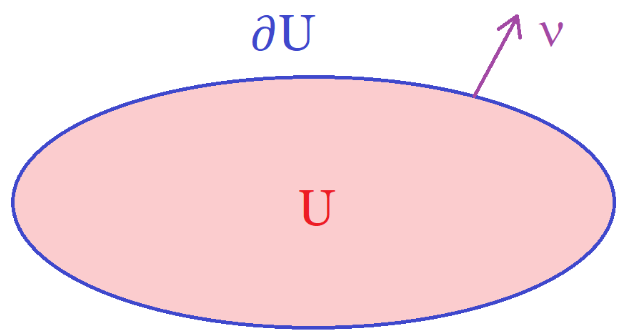
Laplace's Equation:

$$\Delta u = 0$$

Main idea: Laplace's equation measures the density of a fluid in **equilibrium**, and the key ingredient is the:

Divergence Theorem:

$$\int_U \mathrm{div}(F)dx = \int_{\partial U} F \cdot \nu dS$$



Here U is an open and bounded subset of $\mathbb{R}^n$ with boundary ∂U and ν is the unit normal vector to ∂U at a point.

And using this, you eventually get $\Delta u = 0$ (see book for details)

Note: Evans uses notation different from Stewart, and I invite you to look at Appendix A1 of the book if you're even confused about notation. For example, Stewart uses n instead of ν and Stewart's version of the divergence theorem is:

$$\int\int\int_E \operatorname{div}(F) dxdydz = \int\int_S F \cdot n dS$$

4. Applications of Laplace's Equation

Video: Laplace Applications

Laplace's equation is an incredibly powerful PDE, there are a tremendous number of applications:

(1) **Physics:** $u(x)$ gives you the temperature of a metal plate at position x after a really long time (think of a cake that you have let out until it settled)

(2) **Image Processing:** Laplace's equation allows us to go from a pixelated picture to a smooth and rounded/blurry picture

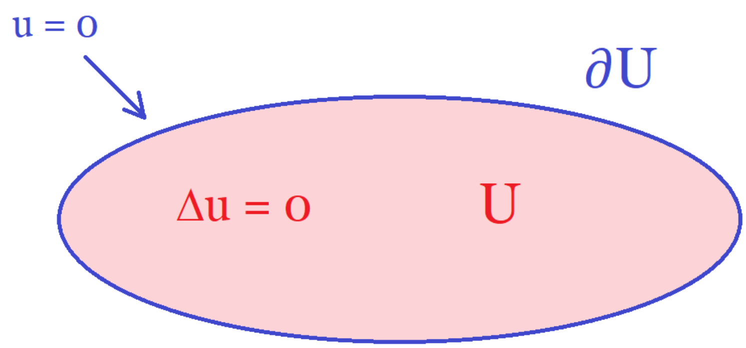
(3) **Complex Analysis:** If $f = f(z) : \mathbb{C} \to \mathbb{C}$ is a holomorphic/analytic function, then both $Re(f)$ and $Im(f)$ solve Laplace's equation.

Example: Take $f(z) = z^2 = (x+iy)^2 = x^2 - y^2 + i(2xy)$. From the above, we can deduce that $Re(f) = x^2 - y^2$ and $Im(f) = 2xy$ solve Laplace's equation

(4) **Why harmonic?** It comes from music!

Let U be the surface of a drum and λ a real number. Consider the solutions of

$$\begin{cases} -\Delta u = \lambda u & (\text{in } U) \\ u = 0 & (\text{on } \partial U) \end{cases}$$



In *most* cases, the only solution to the above PDE is $u = 0$, except for some exceptional values of λ:

Fact:

There is a sequence of numbers $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$ called eigenvalues (or harmonics) for which the above PDE has a nonzero solution

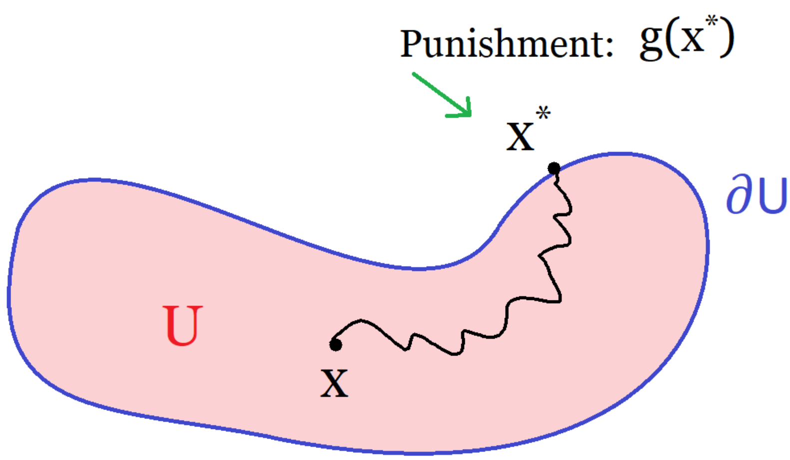
(Compare to eigenvalues in Math 3A: Numbers λ for which $Ax = \lambda x$ has a nonzero solution)

Note: λ_1 is called the principal harmonic, and the other eigenvalues are called the overtones. Those are the numbers you *hear* when someone plays a drum.

This leads to a famous question posed by Marc Kac: *Can you hear the shape of a drum?* In other words, if I *only* tell you what the eigenvalues are (= the sounds you hear), can you tell me what U is? (= the drum).

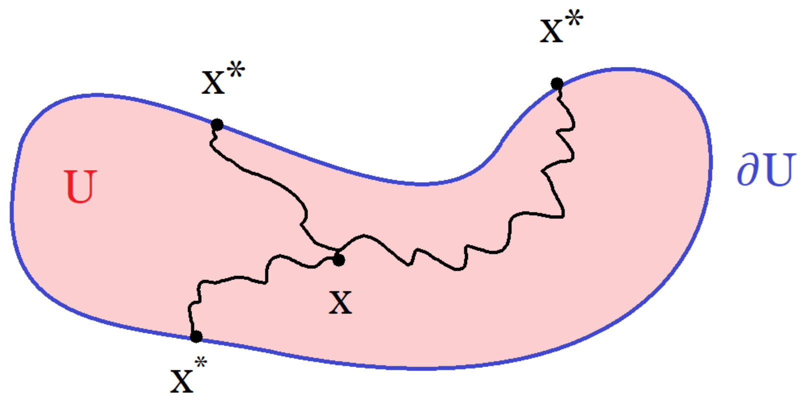
Answer: YES in 2 dimensions if U is smooth (= has no corners), no if U has corners. And generally no in higher dimensions; there is a 16-dimensional counterexample!

(5) **Brownian Motion:** Suppose you start at a point x and you perform Brownian motion until you hit a wall at $x^\star$, where you pay a penalty $g(x^\star)$.



(Analogy: You're driving drunk (please don't do this!), and $g(x^\star)$ is the money that you have to pay to the insurance)

This is a random process, but we can still calculate iyts average value of this event.



Let $u(x)$ = Expected payoff/penalty you get, starting at x.

Cool Fact

Then u solves $\Delta u = 0$

This follows essentially from the mean-value formula for Laplace's equation, which we'll talk about in a couple of weeks.

5. Derivation of the Fundamental Solution

Reading: Section 2.2.1a: Derivation of fundamental solution

Video: Fundamental Solution of Laplace's Equation

Main Take-Away: Since PDEs are really hard to solve, sometimes you'll have to guess that the solution has a special form. Here it turns out that Laplace's equation is invariant under rotations (see Problem 2 in the book, which has solutions on my website).

Here it's useful to look for solutions that are invariant under rotations, that is for radial solutions: $u(x) = v(|x|)$ where $v = v(r)$ (function of one variable only). The rest is just a matter of plugging in your solution in your PDE and using the Chen Lu (Chain Rule)

From that you obtain the Fundamental Solution:

Fundamental Solution:

$$\Phi(x) = \frac{C}{|x|^{n-2}}$$

(In ≥ 3 dimensions, where C is a specific constant that depends only on the dimension; in 2 dimensions, the solution looks like $-\ln$)