LECTURE 23: CURL

The next 3 lectures is all about what happens in 3 dimensions.

1. Motivation

Recall: In 2 dimensions, the quantity $Q_x - P_y$ was useful:

(1) To check if F is conservative

(2) For Green's Theorem

Goal

What is the analog of $Q_x - P_y$ in 3 dimensions?

Suppose $F = \langle P, Q, R \rangle$ is conservative, that is $F = \nabla f$

$$\text{Then:} \quad \langle P, Q, R \rangle = \nabla f = \langle f_x, f_y, f_z \rangle$$

So by Clairaut:

$$f_{xy} = f_{yx} \Rightarrow (f_x)_y = (f_y)_x \Rightarrow P_y = Q_x \Rightarrow Q_x - P_y = 0$$

$$f_{yz} = f_{zy} \Rightarrow (f_y)_z = (f_z)_y \Rightarrow Q_z = R_y \Rightarrow R_y - Q_z = 0$$

Date: Wednesday, March 4, 2020.

$$f_{xz} = f_{zx} \Rightarrow (f_x)_z = (f_z)_x \Rightarrow P_z = R_x \Rightarrow P_z - R_x = 0$$

The amazing thing is that there is **one** operation that takes care of all **three** cases at once.

2. Curl

Definition

$$\text{curl}(F) = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$$

Example 1:

Find $\text{curl}(F)$, where $F = \langle 0, -z, y \rangle$

$$\begin{aligned} \text{curl}(F) &= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & -z & y \end{vmatrix} \\ &= \left\langle \frac{\partial}{\partial y}(y) - \frac{\partial}{\partial z}(-z), -\frac{\partial}{\partial x}(y) + \frac{\partial}{\partial z}(0), \frac{\partial}{\partial x}(-z) - \frac{\partial}{\partial y}(0) \right\rangle \\ &= \langle 1+1, 0, 0 \rangle \\ &= \langle 2, 0, 0 \rangle \end{aligned}$$

Remark: curl(F) is a vector, not a number! (as opposed to div(F), which is a number)

Example 2:

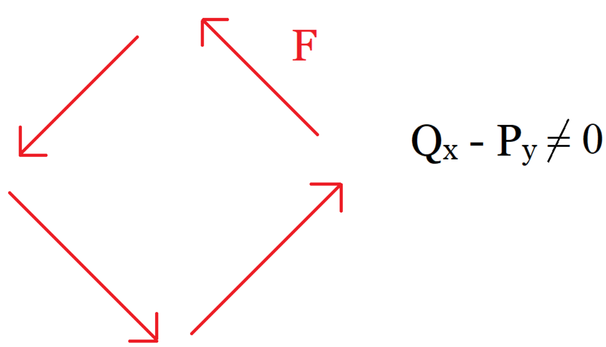
Find curl F, where $F = \langle xz, yz, xy \rangle$

$$\begin{aligned} \text{curl}(F) &= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & yz & xy \end{vmatrix} \\ &= \left\langle \frac{\partial}{\partial y}(xy) - \frac{\partial}{\partial z}(yz), -\frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial z}(xz), \frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial y}(xz) \right\rangle \\ &= \langle x - y, -y + x, 0 \rangle \end{aligned}$$

3. Interpretation

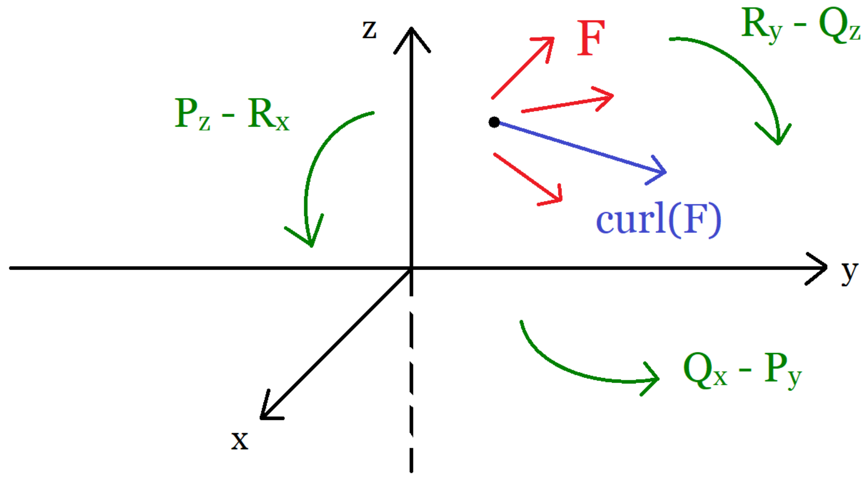
Intuitively: curl(F) measures the **rotation** of F.

Recall: In 2 dimensions, $Q_x - P_y$ measures the microscopic rotation of F



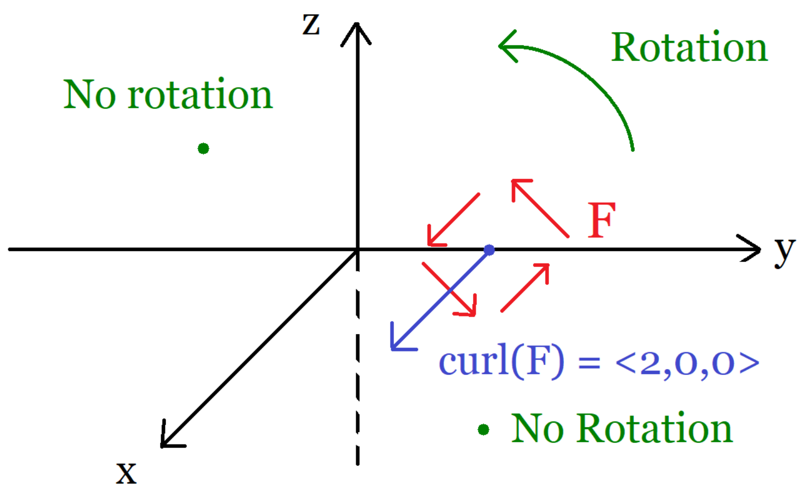
Here we have a 3 dimensional version of this phenomenon:

$$\text{curl}(F) = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$$



So curl(F) measures how F rotates, but in each plane.

Example: $F = \langle 0, -z, y \rangle$, showed curl$(F) = \langle 2, 0, 0 \rangle$



Here curl(F) is the axis of rotation of F

4. Conservative Vector Fields

The most important thing about curl is that it gives us a very elegant way of checking whether a vector field is conservative or not.

Fact:

If F is conservative, then curl$(F) = \langle 0, 0, 0\rangle$

Why? At the beginning of lecture, we showed that if $F = \langle P, Q, R\rangle$ is conservative, then

$$\begin{cases} Q_x - P_y = 0 \\ R_y - Q_z = 0 \\ P_z - R_x = 0 \end{cases}$$

Therefore:

$$\text{curl}(F) = \left\langle \underbrace{R_y - Q_z}_{0}, \underbrace{P_z - R_x}_{0}, \underbrace{Q_x - P_y}_{0} \right\rangle = \langle 0, 0, 0\rangle$$

Conversely: If curl$(F) = \langle 0, 0, 0\rangle$ (and no holes), then F is conservative (will prove this later)

Important Fact:

$$F \text{ conservative } \Leftrightarrow \text{curl}(F) = \langle 0, 0, 0\rangle$$

(So this is a good test for conservative in 3 dimensions, 3D analog of $P_y = Q_x$)

Interpretation: Conservative vector fields are **irrotational** (curl is 0), just like in 2 dimensions.

Example 3:

(a) Is $F = \left\langle y^2z^3, 2xyz^3, 3xy^2z^2 \right\rangle$ conservative?

$$\begin{aligned}
\operatorname{curl}(F) &= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix} \\
&= \left\langle \frac{\partial}{\partial y}(3xy^2z^2) - \frac{\partial}{\partial z}(2xyz^3), -\frac{\partial}{\partial x}(3xy^2z^2) + \frac{\partial}{\partial z}(y^2z^3), \right. \\
&\quad \left. \frac{\partial}{\partial x}(2xyz^3) - \frac{\partial}{\partial y}(y^2z^3) \right\rangle \\
&= \left\langle \cancel{6xyz^2} - \cancel{6xyz^2}, \cancel{-3y^2z^2} + \cancel{3y^2z^2}, \cancel{2yz^3 - 2yz^3} \right\rangle \\
&= \langle 0, 0, 0 \rangle \qquad \textcolor{red}{BINGO!}
\end{aligned}$$

Answer: Yes

(b) Find f such that $F = \nabla f$

$$\left\langle y^2z^3, 2xyz^3, 3xy^2z^2 \right\rangle = \langle f_x, f_y, f_z \rangle$$

$$
\begin{aligned}
f_x = y^2 z^3 &\Rightarrow f = xy^2z^3 + \text{ JUNK} \\
f_y = 2xyz^3 &\Rightarrow f = xy^2z^3 + \text{ JUNK} \\
f_z = 2xy^2z^2 &\Rightarrow f = xy^2z^3 + \text{ JUNK}
\end{aligned}
$$

$$f(x, y, z) = xy^2z^3$$

(c) Evaluate $\int_C F \cdot dr$, C any path connecting $(0, 0, 0)$ and $(1, 1, 1)$

$$
\begin{aligned}
\int_C F \cdot dr =& f(1,1,1) - f(0,0,0) \\
=& (1)(1^2)(1^3) - (0)(0^2)(0^3) \\
=& 1
\end{aligned}
$$

5. Div, Grad, Curl

Recall:

If $F = \langle P, Q, R \rangle$, then $\text{div}(F) = P_x + Q_y + R_z$

Question: How are $\text{div}(F), \nabla f, \text{curl}(F)$ related?

Important Facts:

(1) $\text{curl}(\nabla f) = \langle 0, 0, 0 \rangle$

(2) $\text{div}(\text{curl}(F)) = 0$

Why?

(1) Direct calculation, or: ∇f is conservative, so $\text{curl}(\nabla f) = \langle 0, 0, 0 \rangle$ (by fact above)

(2) Direct calculation

Mnemonic: If you follow the book's order, then

$$\text{New topic} \ \left(\text{Topic before that} \right) = 0$$

$$\underbrace{\text{curl}}_{16.5} \underbrace{\nabla f}_{14.6} = \langle 0, 0, 0 \rangle$$

$$\underbrace{\text{div}}_{16.5 \text{ Part } 1} \ \underbrace{\text{curl}\, F}_{16.5 \text{ Part } 2} = 0$$

Example 4:

Can $F = \left\langle xz, xyz, -y^2 \right\rangle$ be written as $\operatorname{curl} G$ for some G?

No! Suppose $F = \operatorname{curl} G$, then

$$\operatorname{div}(F) = \operatorname{div}(\operatorname{curl} G) = 0 \qquad \text{(By Fact)}$$

$$\text{But:} \quad \operatorname{div}(F) = (xz)_x + (xyz)_y + (-y^2)_z = z + xz \neq 0$$

So $0 \neq 0$, which is a (juicy) contradiction

Warning:

$$\operatorname{div}(\nabla f) \neq 0$$

In fact, $\operatorname{div}(\nabla f) = \Delta f$ (from last time)

Joke: Why is Harvard well-suited for vector calculus? Because the grad (= graduate) school is next to the div (= divinity) school!