LECTURE 24: STOKES’ THEOREM (I)

Welcome to our fourth and final FTC for vector fields, which you can really think of Green’s theorem, but in 3 dimensions.

1. Stokes’ Theorem

Motivation:

\[2B: \int \int F'' = \int F\]

Stokes’ Theorem

Let \( S \) be a surface with boundary \( C \), then:

\[ \int \int_S \text{curl}(F) \cdot dS = \int_C F \cdot dr \]
Today: We’ll use Stokes to calculate $\int \int_S \text{curl}(F)$, and next time we’ll use Stokes to calculate $\int_C F \cdot dr$

2. Example

Example 1:

Evaluate $\int \int_S \text{curl}(F) \cdot dS$

$F = \langle xz, y^2, xy \rangle$

$S$ is the paraboloid $z = 1 - x^2 - y^2$ above the $xy$–plane.

(1) Picture:
Warning: Make sure that the orientation of $C$ (clockwise/counterclockwise) matches with that of $S$ (upwards/outwards)

Trick: If you’re walking on $C$ with your head in the direction of $\hat{n}$ (think upwards), then $S$ should be to your LEFT

Mnemonic: WALK LEFT

Here: $C$ is counterclockwise (90 percent of the time it is)

(2) By Stokes:

$$\int \int_S \text{curl}(F) \cdot dS = \int_C F \cdot dr$$

Hard \hspace{1cm} Easier

$C$ is a circle of radius 1

(because $z = 1 - x^2 - y^2$ and $z = 0$ gives $x^2 + y^2 = 1$)

Parametrize $C$: $r(t) = \langle \cos(t), \sin(t), 0 \rangle, 0 \leq t \leq 2\pi$
\[
\int_C F \cdot dr = \int_0^{2\pi} F(r(t)) \cdot r'(t) dt \\
= \int_0^{2\pi} \langle \cos(t)(0), \sin^2(t), \cos(t) \sin(t) \rangle \cdot \langle -\sin(t), \cos(t), 0 \rangle r'(t) dt \\
= \int_0^{2\pi} \sin^2(t) \cos(t) dt \\
= \left[ \frac{1}{3} \sin^3(t) \right]_0^{2\pi} \\
= 0
\]

3. Intuition
Recall: Green’s Theorem
\[
\int \int_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dx dy = \int_C F \cdot dr \\
\text{Sum of Micro-Rotations} \quad \text{Macro Circulation}
\]
Stokes is nothing other than a 3D analog of Green!

**Recall:** curl($F$) measures the rotation of $F$

\[
\int_S \text{curl}(F) \cdot dS = \int_C F \cdot dr
\]

**Stokes:**

\[
\text{Sum of Micro Rotations} = \text{Macro Circulation}
\]

So Stokes is really just a curvy analog of Green (alternatively: Green is a flat analog of Stokes)

**Analogy:** Suppose you want to count the number of cars in a parking lot. You could either walk around the parking lot and count all the cars ($\int_C F \cdot dr$) or you could walk inside the lot and count how many
cars go in and out of the lot ($\int \int_S F \cdot dS$)

4. Orientation

Video: Integral over a Barrel

Example 2: (Tricky!)

Evaluate $\int \int_S \text{curl}(F) \cdot dS$

$F = \langle yz, -xz, e^z \rangle$

$S$ is the portion of the sphere $x^2 + y^2 + z^2 = 25$ with $-4 < z < 4$ (without the top and bottom)

(1) Picture:
Warning:

(i) Here $C$ has 2 pieces: $C_1$ and $C_2$

(ii) Beware of the orientation! Since you want $S$ to be on your left (Walk Left), $C_1$ has to be clockwise and $C_2$ has to be counterclockwise (reversed)

(2) By Stokes:

$$\int \int_S \text{curl}(F) \cdot dS = \int_C F \cdot dr = \int_{C_1} F \cdot dr + \int_{C_2} F \cdot dr$$

(3) $\int_{C_2} F \cdot dr$ (easier)
Since $z = -4$ on $C_2$, we get

\[ x^2 + y^2 + z^2 = 25 \Rightarrow x^2 + y^2 + (-4)^2 = 25 \Rightarrow x^2 + y^2 = 25 - 16 = 9 \]

So $C_2$ is a circle of radius 3, with $z = -4$, counterclockwise

\[
 r(t) = \langle 3 \cos(t), 3 \sin(t), -4 \rangle \quad 0 \leq t \leq 2\pi
\]

\[
 \int_{C_2} F \cdot dr = \int_0^{2\pi} \left(3 \sin(t)(-4), -3 \cos(t)(-4), e^{-4}\right) \cdot \left(-3 \sin(t), 3 \cos(t), 0\right) dt
\]

\[
 = \int_0^{2\pi} \frac{36 \sin^2(t) + 36 \cos^2(t)}{36} dt
\]

\[
 = 36(2\pi)
\]

\[
 = 72\pi
\]

\[ (4) \int_{C_1} F \cdot dr \]

Since $z = 4$ on $C_1$, we get

\[ x^2 + y^2 + z^2 = 25 \Rightarrow x^2 + y^2 + 4^2 = 25 \Rightarrow x^2 + y^2 = 9 \]

$C_1$ is a circle of radius 3, with $z = 4$, but in the clockwise direction
“Parametrize” $C_1$:

$$r(t) = \langle 3 \cos(t), 3 \sin(t), 4 \rangle \quad (0 \leq t \leq 2\pi)$$

$$\int_{C_1} F \cdot dr = -\int_0^{2\pi} \underbrace{\langle 3 \sin(t)(4), -3 \cos(t)(4), e^4 \rangle}_{\langle y^2, -xz, e^z \rangle} \cdot \langle -3 \sin(t), 3 \cos(t), 0 \rangle \, dt$$

$$= -\int_0^{2\pi} \underbrace{-36 \sin^2(t) + -36 \cos^2(t)}_{-36} \, dt$$

$$= 36(2\pi)$$

$$= 72\pi$$

(5) Conclusion:

$$\int_C F \cdot dr = \int_{C_1} F \cdot dr + \int_{C_2} F \cdot dr = 72\pi + 72\pi = 144\pi$$
Alternate Solution: (Courtesy Andreas Haberstroh)

Let’s evaluate this integral using the divergence theorem trick, by closing the surface.

(1) Let $S_1$ be the top disk and $S_2$ be the bottom disk, as in the following picture:

Then $S + S_1 + S_2$ is closed, so by the divergence theorem:
\[
\int \int_{S+S_1+S_2} \text{curl}(F) \cdot d\mathbf{S} = \int \int \int_{E} \text{div}(\text{curl}(F)) \, dxdydz
\]
\[
= \int \int \int_{E} 0
\]
\[
= 0
\]

Here we used the fact that \(\text{div}(\text{curl}(F)) = 0\)

Therefore:

\[
\int \int_{S} \text{curl}(F) \cdot d\mathbf{S} + \int \int_{S_1} \text{curl}(F) \cdot d\mathbf{S} + \int \int_{S_2} \text{curl}(F) \cdot d\mathbf{S} = 0
\]

Hence:

\[
\int \int_{S} \text{curl}(F) \cdot d\mathbf{S} = -\int \int_{S_1} \text{curl}(F) \cdot d\mathbf{S} - \int \int_{S_2} \text{curl}(F) \cdot d\mathbf{S}
\]

(3) \(\int_{S_1} \text{curl}(F) \cdot d\mathbf{S}\)

\[
\text{curl}(F) = \begin{vmatrix}
i & j & k \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
yz & -xz & e^z
\end{vmatrix}
\]

\[
= \left( \frac{\partial}{\partial y}(e^z) + \frac{\partial}{\partial z}(xz), -\frac{\partial}{\partial x}(e^z) + \frac{\partial}{\partial z}(yz), \frac{\partial}{\partial x}(-xz) - \frac{\partial}{\partial y}(yz) \right)
\]

\[
= \langle x, y, -2z \rangle
\]
Parametrize $S_1$: $r(x, y) = \langle x, y, 4 \rangle$ (since $z = 4$)

$r_x = \langle 1, 0, 0 \rangle, r_y = \langle 0, 1, 0 \rangle$

\[
\hat{n} = \begin{vmatrix} i & j & k \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = \langle 0, 0, 1 \rangle \quad \text{(Points up)}
\]

\[
\int \int_{S_1} \text{curl}(F) \cdot dS = \int \int_{D} \langle x, y, -8 \rangle \cdot (0, 0, 1) \, dx \, dy
\]

\[
= \int \int_{D} -8 \, dx \, dy
\]

\[
= -8 (\text{Area}(D))
\]

\[
= -8\pi(3^2) \quad \text{(D is a disk of radius 3)}
\]

\[
= -72\pi
\]

(4) $\int_{S_2} \text{curl}(F) \cdot dS$

\[
\text{curl}(F) = \langle x, y, -2z \rangle
\]

Parametrize $S_2$: $r(x, y) = \langle x, y, -4 \rangle$ (since $z = -4$)

$r_x = \langle 1, 0, 0 \rangle, r_y = \langle 0, 1, 0 \rangle$

\[
\hat{n} = \begin{vmatrix} i & j & k \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = \langle 0, 0, 1 \rangle
\]
But since we want $\hat{n}$ to point \textbf{down} (see picture), we need to choose $\hat{n} = \langle 0, 0, -1 \rangle$

\[
\iint_{S_2} \text{curl}(F) \cdot d\mathbf{S} = \int \int_D \langle x, y, 8 \rangle \cdot \langle 0, 0, -1 \rangle \, dxdy
\]

\[= \int \int_D -8dxdy \]

\[= -8(Area(D)) \]

\[= -8\pi(3^2) \quad (D \text{ is a disk of radius } 3) \]

\[= -72\pi \]

\[\text{(5) Answer} \]

\[
\iint_S \text{curl}(F) \cdot d\mathbf{S} = - \iint_{S_1} \text{curl}(F) \cdot d\mathbf{S} - \iint_{S_2} \text{curl}(F) \cdot d\mathbf{S} \\
\]

\[= -(-72\pi) - (-72\pi) \]

\[= 144\pi \]