

LECTURE 24: STOKES' THEOREM (I)

Welcome to our fourth and final FTC for vector fields, which you can really think of Green's theorem, but in 3 dimensions.

1. STOKES' THEOREM

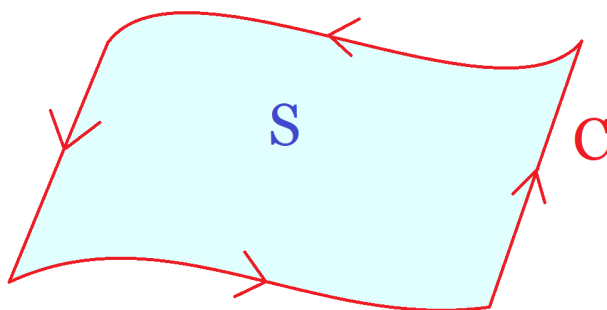
Motivation:

$$2B: \int \int F' = \int F$$

Stokes' Theorem

Let S be a surface with boundary C , then:

$$\int \int_S \text{curl}(F) \cdot d\mathbf{S} = \int_C F \cdot dr$$



Today: We'll use Stokes to calculate $\int \int_S \text{curl}(F)$, and next time we'll use Stokes to calculate $\int_C F \cdot dr$

2. EXAMPLE

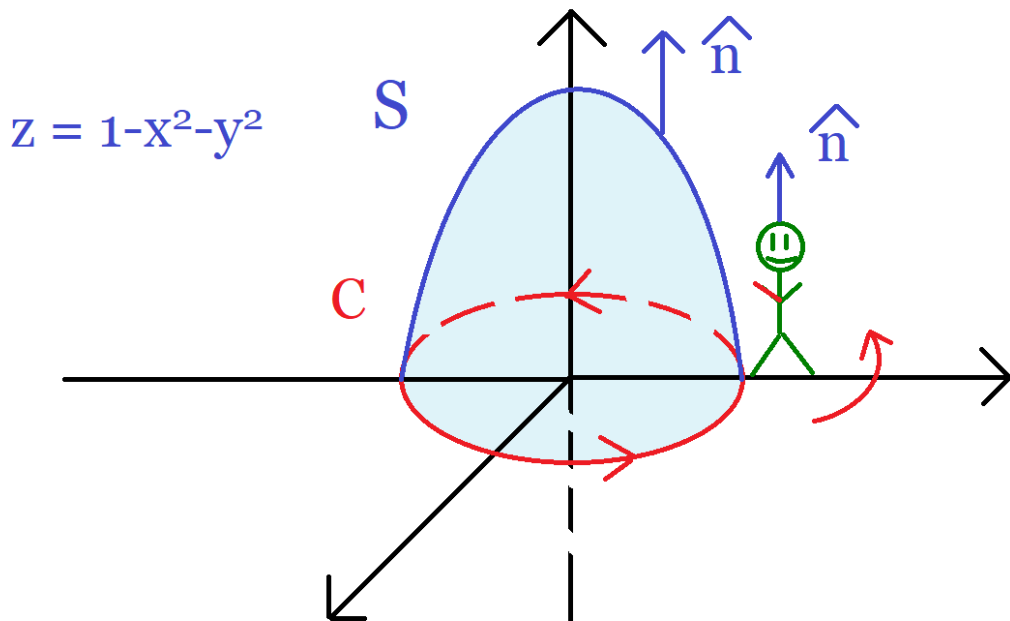
Example 1:

Evaluate $\int \int_S \text{curl}(F) \cdot d\mathbf{S}$

$$F = \langle xz, y^2, xy \rangle$$

S is the paraboloid $z = 1 - x^2 - y^2$ above the xy -plane.

(1) **Picture:**



Warning: Make sure that the orientation of C (clockwise/counterclockwise) matches with that of S (upwards/outwards)

Trick: If you're walking on C with your head in the direction of \hat{n} (think upwards), then S should be to your **LEFT**

Mnemonic: WALK LEFT

Here: C is counterclockwise (90 percent of the time it is)

(2) By Stokes:

$$\underbrace{\int \int_S \text{curl}(F) \cdot d\mathbf{S}}_{\text{Hard}} = \underbrace{\int_C F \cdot dr}_{\text{Easier}}$$

C is a circle of radius 1

(because $z = 1 - x^2 - y^2$ and $z = 0$ gives $x^2 + y^2 = 1$)

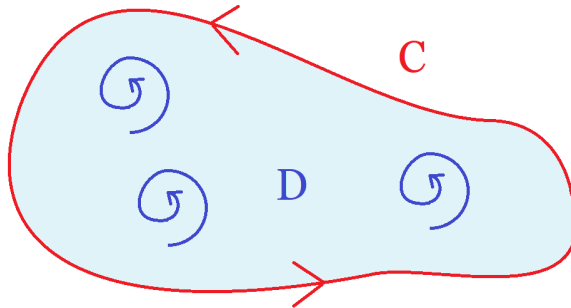
Parametrize C : $r(t) = \langle \cos(t), \sin(t), 0 \rangle, 0 \leq t \leq 2\pi$

$$\begin{aligned}
\int_C F \cdot dr &= \int_0^{2\pi} F(r(t)) \cdot r'(t) dt \\
&= \int_0^{2\pi} \underbrace{\langle \cos(t)(0), \sin^2(t), \cos(t) \sin(t) \rangle}_{\langle xz, y^2, xy \rangle} \cdot \underbrace{\langle -\sin(t), \cos(t), 0 \rangle}_{r'(t)} dt \\
&= \int_0^{2\pi} \sin^2(t) \cos(t) dt \\
&= \left[\frac{1}{3} \sin^3(t) \right]_0^{2\pi} \\
&= 0
\end{aligned}$$

3. INTUITION

Recall: Green's Theorem

$$\underbrace{\int \int_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dx dy}_{\text{Sum of Micro-Rotations}} = \underbrace{\int_C F \cdot dr}_{\text{Macro Circulation}}$$

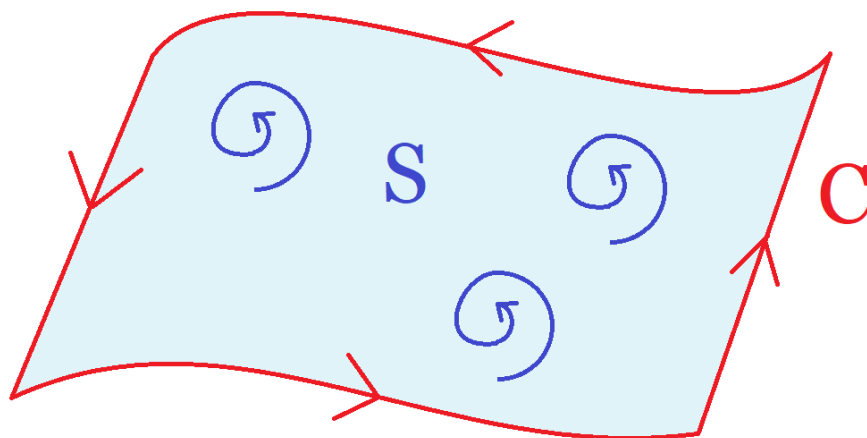


Stokes is nothing other than a 3D analog of Green!

Recall: $\text{curl}(F)$ measures the rotation of F

Stokes:

$$\underbrace{\int \int_S \text{curl}(F) \cdot d\mathbf{S}}_{\text{Sum of Micro Rotations}} = \underbrace{\int_C F \cdot d\mathbf{r}}_{\text{Macro Circulation}}$$



So Stokes is really just a curvy analog of Green (alternatively: Green is a flat analog of Stokes)

Analogy: Suppose you want to count the number of cars in a parking lot. You could either walk around the parking lot and count all the cars ($\int_C F \cdot d\mathbf{r}$) or you could walk inside the lot and count how many

cars go in and out of the lot ($\int \int_S F \cdot d\mathbf{S}$)

4. ORIENTATION

Video: Integral over a Barrel

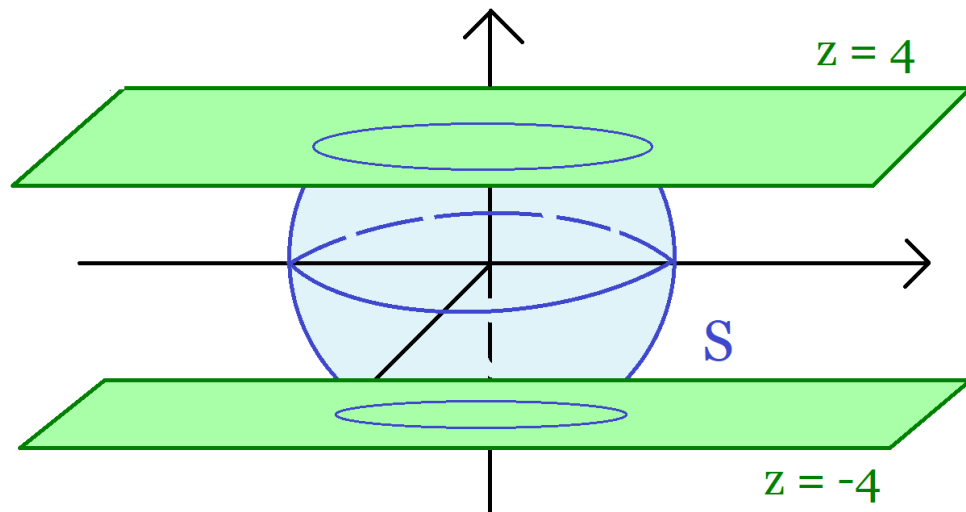
Example 2: (Tricky!)

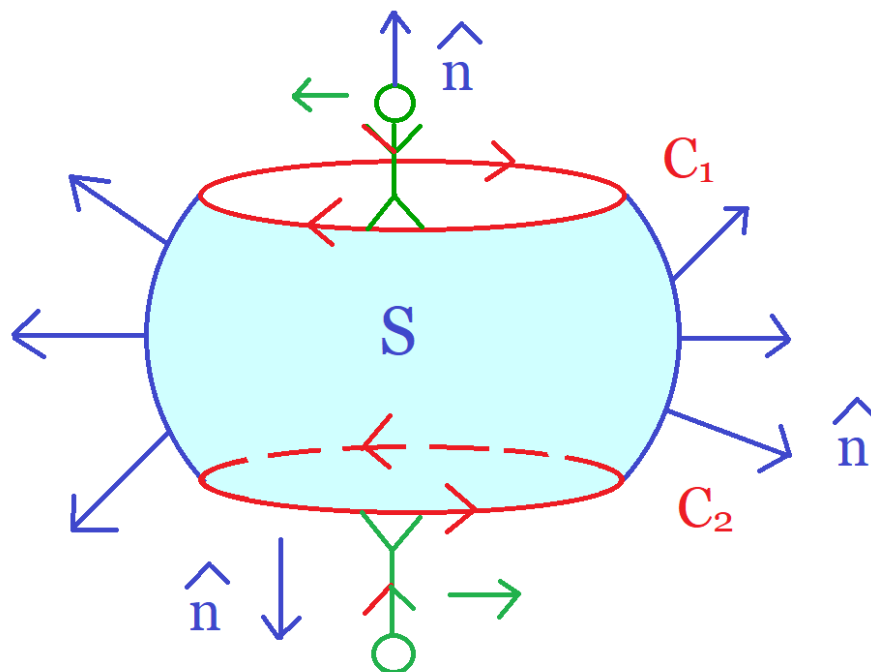
Evaluate $\int \int_S \text{curl}(F) \cdot d\mathbf{S}$

$$F = \langle yz, -xz, e^z \rangle$$

S is the portion of the sphere $x^2 + y^2 + z^2 = 25$ with $-4 < z < 4$ (without the top and bottom)

(1) **Picture:**





Warning:

- (i) Here C has 2 pieces: C_1 and C_2
 - (ii) Beware of the orientation! Since you want S to be on your left (Walk Left), C_1 has to be **clockwise** and C_2 has to be counterclockwise (reversed)
- (2) By Stokes:

$$\int \int_S \text{curl}(F) \cdot d\mathbf{S} = \int_C F \cdot dr = \int_{C_1} F \cdot dr + \int_{C_2} F \cdot dr$$

- (3) $\int_{C_2} F \cdot dr$ (easier)

Since $z = -4$ on C_2 , we get

$$x^2 + y^2 + z^2 = 25 \Rightarrow x^2 + y^2 + (-4)^2 = 25 \Rightarrow x^2 + y^2 = 25 - 16 = 9$$

So C_2 is a circle of radius 3, with $z = -4$, counterclockwise

$$r(t) = \langle 3 \cos(t), 3 \sin(t), -4 \rangle \quad 0 \leq t \leq 2\pi$$

$$\begin{aligned} \int_{C_2} F \cdot dr &= \int_0^{2\pi} \underbrace{\langle 3 \sin(t)(-4), -3 \cos(t)(-4), e^{-4} \rangle}_{\langle yz, -xz, e^z \rangle} \cdot \langle -3 \sin(t), 3 \cos(t), 0 \rangle dt \\ &= \int_0^{2\pi} \underbrace{36 \sin^2(t) + 36 \cos^2(t)}_{36} dt \\ &= 36(2\pi) \\ &= 72\pi \end{aligned}$$

$$(4) \int_{C_1} F \cdot dr$$

Since $z = 4$ on C_1 , we get

$$x^2 + y^2 + z^2 = 25 \Rightarrow x^2 + y^2 + 4^2 = 25 \Rightarrow x^2 + y^2 = 9$$

C_1 is a circle of radius 3, with $z = 4$, but in the **clockwise** direction

“Parametrize” C_1 :

$$r(t) = \langle 3 \cos(t), 3 \sin(t), 4 \rangle \quad (0 \leq t \leq 2\pi)$$

$$\begin{aligned} \int_{C_1} F \cdot dr &= - \int_0^{2\pi} \underbrace{\langle 3 \sin(t)(4), -3 \cos(t)(4), e^4 \rangle}_{\langle yz, -xz, e^z \rangle} \cdot \langle -3 \sin(t), 3 \cos(t), 0 \rangle dt \\ &= - \int_0^{2\pi} \underbrace{-36 \sin^2(t) + -36 \cos^2(t)}_{-36} dt \\ &= 36(2\pi) \\ &= 72\pi \end{aligned}$$

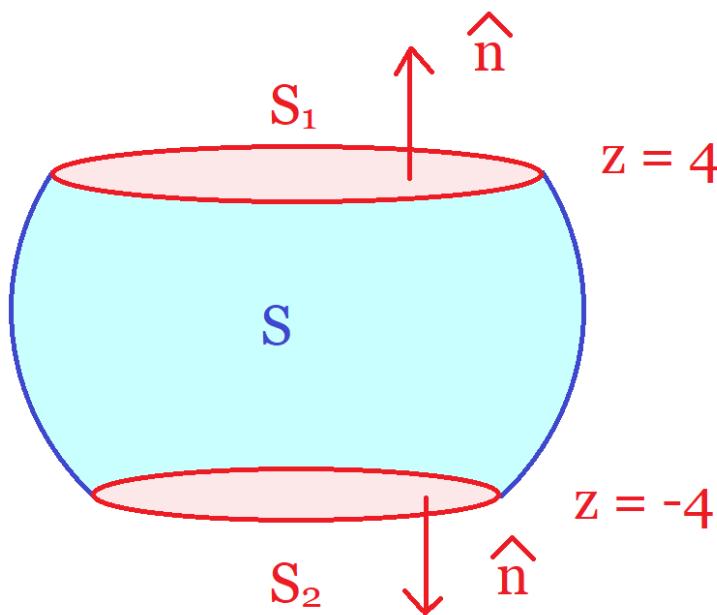
(5) Conclusion:

$$\int_C F \cdot dr = \int_{C_1} F \cdot dr + \int_{C_2} F \cdot dr = 72\pi + 72\pi = 144\pi$$

Alternate Solution: (Courtesy Andreas Haberstroh)

Let's evaluate this integral using the divergence theorem trick, by closing the surface.

- (1) Let S_1 be the top disk and S_2 be the bottom disk, as in the following picture:



Then $S + S_1 + S_2$ is closed, so by the divergence theorem:

$$\begin{aligned}
\int \int_{S+S_1+S_2} \text{curl}(F) \cdot d\mathbf{S} &= \int \int \int_E \text{div}(\text{curl}(F)) dx dy dz \\
&= \int \int \int_E 0 \\
&= 0
\end{aligned}$$

Here we used the fact that $\text{div}(\text{curl}(F)) = 0$

Therefore:

$$\int \int_S \text{curl}(F) \cdot d\mathbf{S} + \int \int_{S_1} \text{curl}(F) \cdot d\mathbf{S} + \int \int_{S_2} \text{curl}(F) \cdot d\mathbf{S} = 0$$

Hence:

$$\int \int_S \text{curl}(F) \cdot d\mathbf{S} = - \int \int_{S_1} \text{curl}(F) \cdot d\mathbf{S} - \int \int_{S_2} \text{curl}(F) \cdot d\mathbf{S}$$

$$(3) \int_{S_1} \text{curl}(F) \cdot d\mathbf{S}$$

$$\begin{aligned}
\text{curl}(F) &= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & -xz & e^z \end{vmatrix} \\
&= \left\langle \frac{\partial}{\partial y}(e^z) + \frac{\partial}{\partial z}(xz), -\frac{\partial}{\partial x}(e^z) + \frac{\partial}{\partial z}(yz), \frac{\partial}{\partial x}(-xz) - \frac{\partial}{\partial y}(yz) \right\rangle \\
&= \langle x, y, -2z \rangle
\end{aligned}$$

Parametrize S_1 : $r(x, y) = \langle x, y, 4 \rangle$ (since $z = 4$)

$$r_x = \langle 1, 0, 0 \rangle, r_y = \langle 0, 1, 0 \rangle$$

$$\hat{n} = \begin{vmatrix} i & j & k \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = \langle 0, 0, 1 \rangle \quad (\text{Points up})$$

$$\begin{aligned} \int \int_{S_1} \text{curl}(F) \cdot d\mathbf{S} &= \int \int_D \underbrace{\langle x, y, -8 \rangle}_{\langle x, y, -2z \rangle} \cdot \underbrace{\langle 0, 0, 1 \rangle}_{\hat{n}} dx dy \\ &= \int \int_D -8 dx dy \\ &= -8(\text{Area}(D)) \\ &= -8\pi(3^2) \quad (\text{D is a disk of radius 3}) \\ &= -72\pi \end{aligned}$$

$$(4) \int_{S_2} \text{curl}(F) \cdot d\mathbf{S}$$

$$\text{curl}(F) = \langle x, y, -2z \rangle$$

Parametrize S_2 : $r(x, y) = \langle x, y, -4 \rangle$ (since $z = -4$)

$$r_x = \langle 1, 0, 0 \rangle, r_y = \langle 0, 1, 0 \rangle$$

$$\hat{n} = \begin{vmatrix} i & j & k \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = \left\langle 0, 0, \underbrace{1}_{\geq 0} \right\rangle$$

But since we want \hat{n} to point **down** (see picture), we need to choose $\hat{n} = \langle 0, 0, -1 \rangle$

$$\begin{aligned}
 \iint_{S_2} \text{curl}(F) \cdot d\mathbf{S} &= \int \int_D \underbrace{\langle x, y, 8 \rangle}_{\langle x, y, -2z \rangle} \cdot \underbrace{\langle 0, 0, -1 \rangle}_{\hat{n}} dx dy \\
 &= \int \int_D -8 dx dy \\
 &= -8(\text{Area}(D)) \\
 &= -8\pi(3^2) \quad (\text{D is a disk of radius 3}) \\
 &= -72\pi
 \end{aligned}$$

(5) Answer

$$\begin{aligned}
 \iint_S \text{curl}(F) \cdot d\mathbf{S} &= - \iint_{S_1} \text{curl}(F) \cdot d\mathbf{S} - \iint_{S_2} \text{curl}(F) \cdot d\mathbf{S} \\
 &= -(-72\pi) - (-72\pi) \\
 &= 144\pi
 \end{aligned}$$