LECTURE 25: STOKES' THEOREM (II)

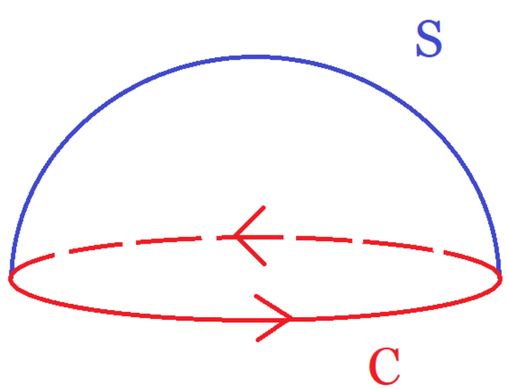
1. Quick Recap

Previously on *I'm so Stoked*, we learned about an amazing theorem that relates line integrals with surface integrals.

Stokes' Theorem

Let S be a surface with boundary C, then:

$$\int\int_S \text{curl}(F) \cdot d\mathbf{S} = \int_C F \cdot dr$$



Date: Tuesday, March 10, 2020.

Last time: We used Stokes to calculate $\int\int_S \operatorname{curl}(F)$, but today we'll use it to calculate $\int_C F \cdot dr$

2. A Cylindrical Example

Video: Stokes' Theorem

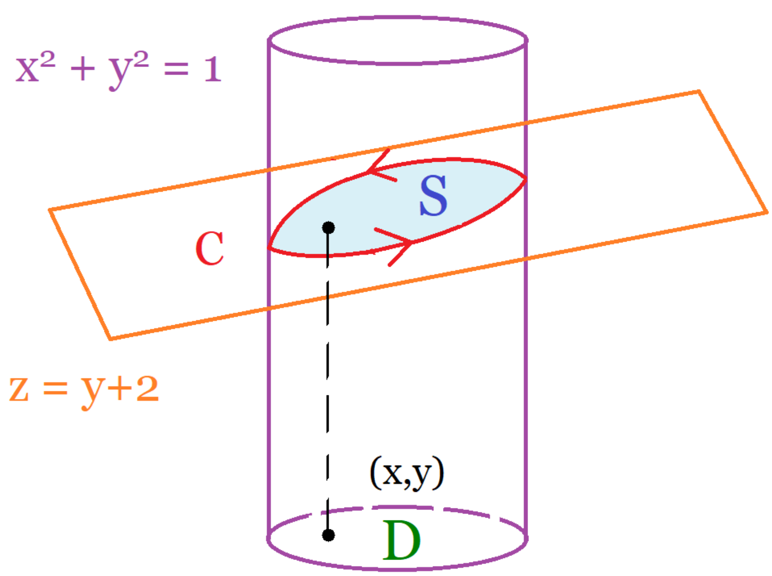
Example 1:

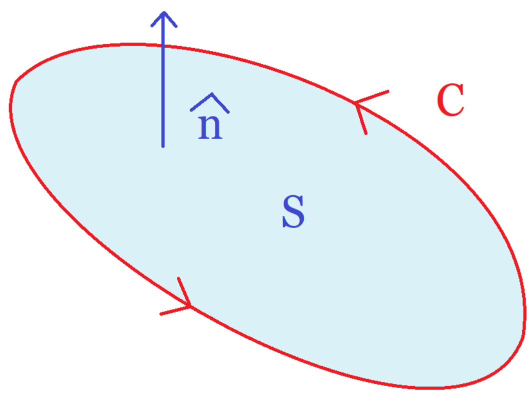
Evaluate $\int_C F \cdot dr$

$F = \langle xy, yz, xz \rangle$

C is the curve of intersection of $z = y + 2$ and $x^2 + y^2 = 1$ (in the counterclockwise direction)

(1) **Picture:**





(2) By Stokes:

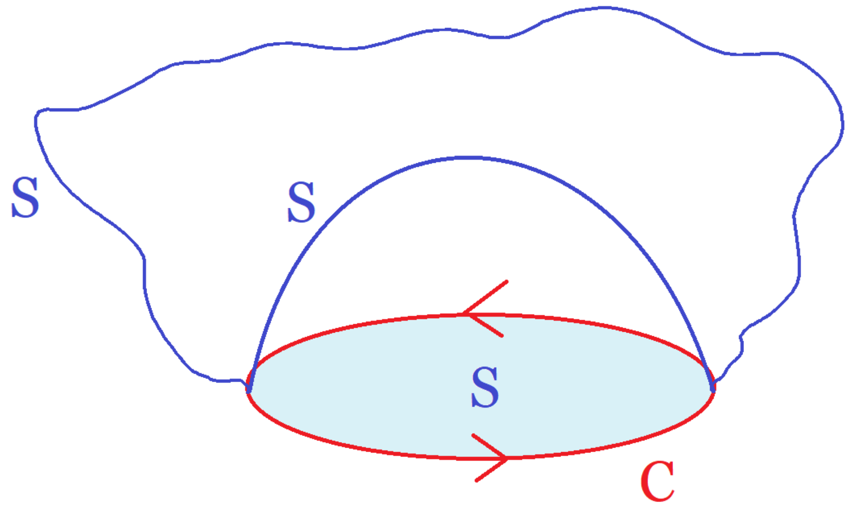
$$\int_C F \cdot dr = \int\int_S \text{curl}(F) \cdot d\mathbf{S}$$

$$\begin{aligned}\text{curl}(F) &= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & xz \end{vmatrix} \\ &= \left\langle \frac{\partial}{\partial y}(xz) - \frac{\partial}{\partial z}(yz), -\frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial z}(xy), \frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial y}(xy) \right\rangle \\ &= \langle -y, -z, -x \rangle \qquad \textit{EASIER}\end{aligned}$$

(3) **What is S?**

Stokes' Miracle

We can choose S to be anything we want, as long as the boundary of S is C



(**Why?** The quantity $\int_C F \cdot dr$ only depends on C, not on S)

Easiest choice: Let S be the interior (inside) of C.

Parametrize S**:**

$$r(x,y) = \left\langle x, y, \underbrace{y+2}_{z} \right\rangle$$

$$r_x = \langle 1, 0, 0 \rangle$$
$$r_y = \langle 0, 1, 1 \rangle$$

$$\hat{n} = r_x \times r_y = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \langle 0, -1, 1 \rangle$$

(Mentally check WALK Left, that is the counterclockwise orientation of C matches the direction of $\hat{n}$)

(4)

$$\begin{aligned}
\int_C F \cdot dr &= \int\int_S \operatorname{curl} F \cdot d\mathbf{S} \\
&= \int\int_D \underbrace{\langle -y, -(y+2), -x\rangle}_{\langle -y,-z,-x\rangle} \cdot \underbrace{\langle 0,-1,1\rangle}_{\hat{n}} dxdy \\
&= \int\int_D y+2-x dxdy \qquad \text{(D is a disk of radius 1)} \\
&= \int_0^{2\pi}\int_0^1 \left(r\sin(\theta)+2-r\cos(\theta)\right) r dr d\theta \\
&= \cdots \\
&= 2\pi
\end{aligned}$$

3. A Spherical Example

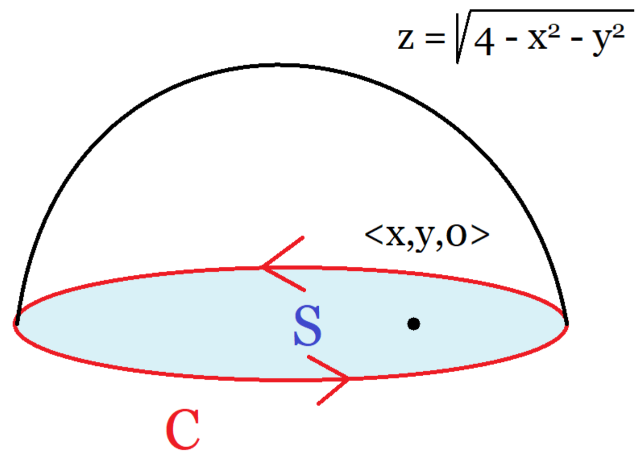
Example 2:

$\int_C F \cdot dr$

$F = \left\langle x+y^2, y+z^2, z+x^2\right\rangle$

C is the boundary of the hemisphere $z = \sqrt{4-x^2-y^2}$ in the $xy-$plane (oriented counterclockwise)

(1) **Picture:**



(2) By Stokes:

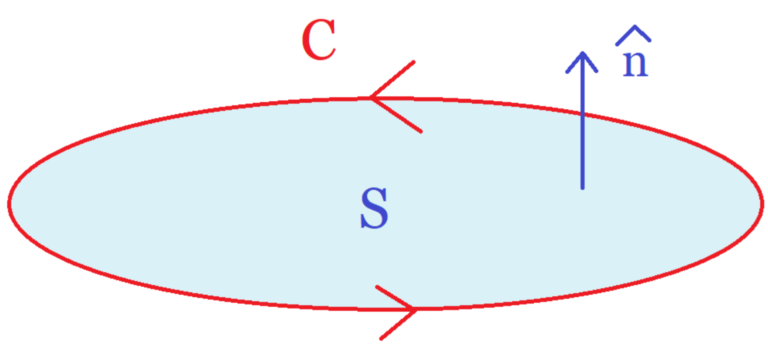
$$\int_C F \cdot dr = \int\int_S \text{curl}(F) \cdot d\mathbf{S}$$

$$\begin{aligned}\text{curl}(F) &= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y^2 & y+z^2 & z+x^2 \end{vmatrix} \\ &= \Big\langle \frac{\partial}{\partial y}(z+x^2) - \frac{\partial}{\partial z}(y+z^2), -\frac{\partial}{\partial x}(z+x^2) + \frac{\partial}{\partial z}(x+y^2), \\ &\quad \frac{\partial}{\partial x}(y+z^2) - \frac{\partial}{\partial y}(x+y^2) \Big\rangle \\ &= \langle -2z, -2x, -2y \rangle \qquad \textit{EASIER}\end{aligned}$$

(3) **What is S?**

Here C is a circle of radius 2 (let $z = 0$ in $z = \sqrt{4 - x^2 - y^2}$ to get $x^2 + y^2 = 4$)

So let $S =$ **DISK** of radius 2 in the $xy-$plane!



$r(x, y) = \langle x, y, 0 \rangle$

$$r_x = \langle 1, 0, 0 \rangle$$
$$r_y = \langle 0, 1, 0 \rangle$$

$$\hat{n} = r_x \times r_y = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \langle 0, 0, 1 \rangle$$

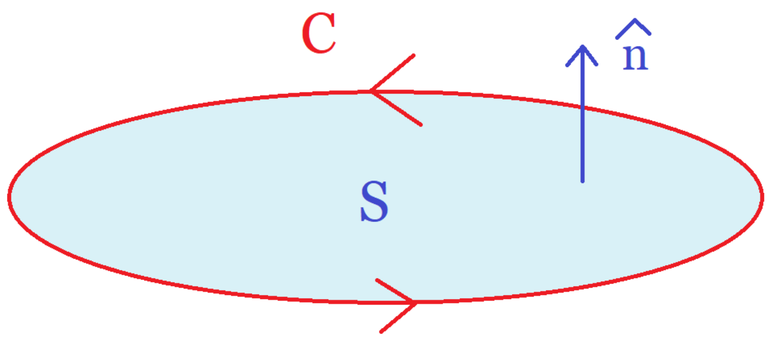
(4)

$$
\begin{aligned}
\int_C F \cdot dr &= \int\int_S \operatorname{curl}(F) \cdot d\mathbf{S} \\
&= \int\int_D \underbrace{\langle 0, -2x, -2y\rangle}_{\langle -2z,-2x,-2y\rangle} \cdot \underbrace{\langle 0,0,1\rangle}_{\hat{n}} dxdy \\
&= \int\int_D -2ydxdy \qquad \text{(D is a disk of radius 2)} \\
&= \int_0^{2\pi}\int_0^2 -2r\sin(\theta) r dr d\theta \\
&= \left(\int_0^2 -2r^2 dr\right)\left(\int_0^{2\pi} \sin(\theta)d\theta\right) \\
&=0
\end{aligned}
$$

4. Concluding Remarks

(1) **Recall:** **IF** $F = \langle P, Q, R\rangle$ is conservative, **THEN** $\operatorname{curl}(F) = \langle 0,0,0\rangle$

Now if $\operatorname{curl}(F) = \langle 0,0,0\rangle$ (and no holes), then for any closed C,

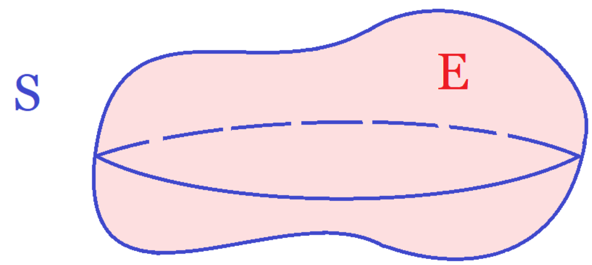


$$\int_C F \cdot dr = \int\int_S \underbrace{\text{curl}(F)}_{\langle 0,0,0 \rangle} \cdot d\mathbf{S} = 0$$

So F is conservative by cute fact from 16.3

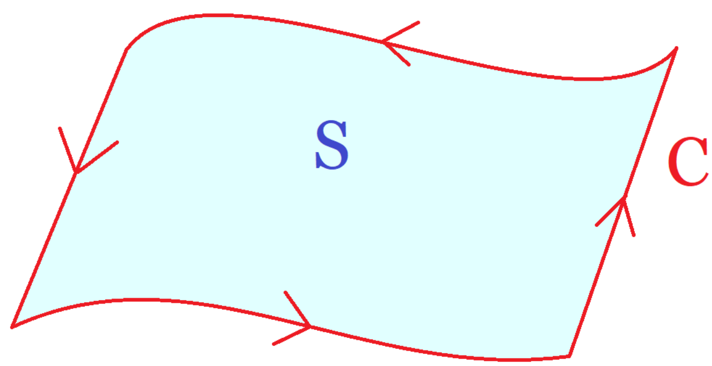
Hence we have: F conservative $\Leftrightarrow \text{curl}(F) = \langle 0, 0, 0 \rangle$

(2) Important: If S is **closed**, then:



$$\int\int_S \mathrm{curl}(F)\cdot d\mathbf{S} \overset{DIV}{=} \int\int\int_E \underbrace{\mathrm{div}(\mathrm{curl}(F))}_{0} = 0$$

Point: Stokes in only interesting for *open* surfaces, where there is a boundary curve C



(3) Stokes FAILS when there is a hole.

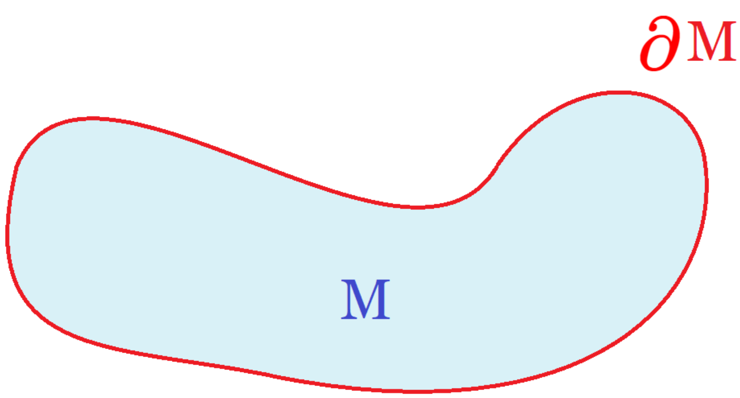
Cohomology Theory: How badly does it fail? Gives us interesting information about the *topology* of the domain.

(4) **AMAZING FACT:**

There is **ONE** theorem that takes care of **ALL** the FTC all at once (FTC Line integrals, Green, Div, Stokes, and even the Math 2B FTC):

General Stokes

$$\int_M d\omega = \int_{\partial M} \omega$$



Here:

$$\begin{aligned}
\partial M &= \text{Boundary of } M \\
\omega &= \text{Function or Vector Field} \\
d\omega &= \text{“Derivative” of } \omega \\
&\quad \text{(Think derivative or curl or div, called a differential form)}
\end{aligned}$$

In every case, you have "Integral of derivative is the Integral of the function over the boundary"