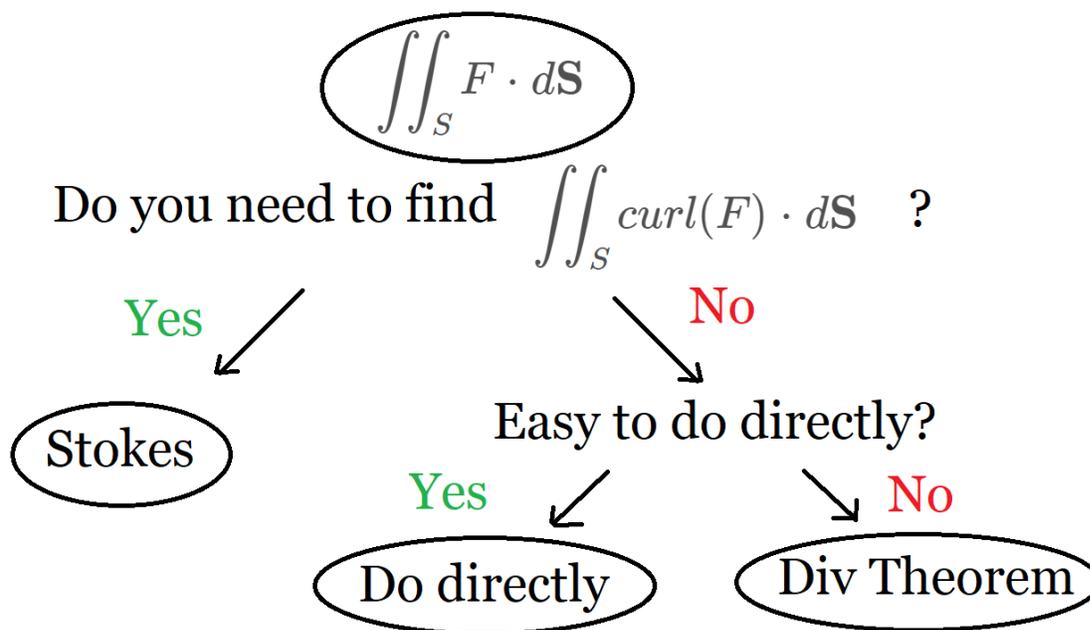


## LECTURE 26: FINAL EXAM REVIEW SESSION (I) - SURFACE INTEGRALS

Welcome to the first part of our final exam review session! Today is all about what to do if you have to calculate a surface integral. Fortunately there is a nice roadmap for what theorem to use when:



### 1. STOKES' THEOREM

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*Date:* Thursday, March 12, 2020.

**Example 1:**

(a) Find  $G$  such that  $G = \text{curl}(F)$ , where  $G = \langle 0, y, -z \rangle$

**Hint:** Guess  $F = \langle P, 0, 0 \rangle$  for some  $P$

$$\begin{aligned} \text{curl}(F) &= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & 0 & 0 \end{vmatrix} \\ &= \left\langle \frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(0), -\frac{\partial}{\partial x}(0) + \frac{\partial}{\partial z}P, \frac{\partial}{\partial x}(0) - \frac{\partial}{\partial y}(P) \right\rangle \\ &= \langle 0, P_z, -P_y \rangle \\ &\stackrel{\text{WANT}}{=} \langle 0, y, -z \rangle \end{aligned}$$

Therefore:

$$\begin{cases} P_z = y \Rightarrow P = \int y dz = yz + \text{JUNK} \\ -P_y = -z \Rightarrow P_y = z \Rightarrow P = \int z dy = zy + \text{JUNK} \end{cases}$$

Hence  $P = zy$ , so

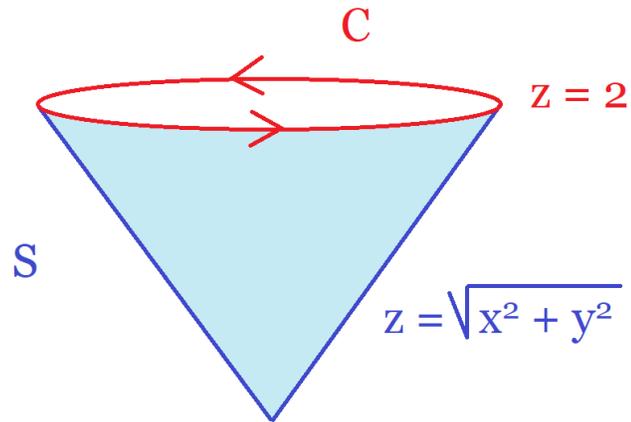
$$F = \langle P, 0, 0 \rangle = \langle zy, 0, 0 \rangle$$

(b)  $\iint_S G \cdot d\mathbf{S}$  (G as in (a))

$S$  is the part of the surface  $z = \sqrt{x^2 + y^2}$  and  $0 \leq z < 2$ .  
Assume  $C$  is oriented counterclockwise

(1) **Picture:**

Notice that  $z = \sqrt{x^2 + y^2} \Rightarrow z^2 = x^2 + y^2$  (Cone)



## (2)

$$\begin{aligned} \iint_S G \cdot d\mathbf{S} &= \iint_S \text{curl}(F) \cdot d\mathbf{S} \quad (\text{By (a)}) \\ &= \int_C F \cdot dr \quad (\text{Stokes}) \end{aligned}$$

Where  $F = \langle yz, 0, 0 \rangle$

(3) **What is  $C$ ?**

$$z = \sqrt{x^2 + y^2} \text{ and } z = 2 \Rightarrow \sqrt{x^2 + y^2} = 2 \Rightarrow x^2 + y^2 = 4$$

So  $C$  is a circle of radius 2, in the clockwise direction

$$r(t) = \langle 2 \cos(t), 2 \sin(t), 2 \rangle, \quad (0 \leq t \leq 2\pi)$$

(4)

$$\begin{aligned} \int_C F \cdot dr &= \int_0^{2\pi} F(r(t)) \cdot r'(t) dt \\ &= \int_0^{2\pi} \underbrace{\langle 2 \sin(t)(2), 0, 0 \rangle}_{\langle yz, 0, 0 \rangle} \cdot \underbrace{\langle -2 \sin(t), 2 \cos(t), 0 \rangle}_{r'(t)} dt \\ &= \int_0^{2\pi} -8 \sin^2(t) dt \\ &= -8 \int_0^{2\pi} \frac{1}{2} - \frac{1}{2} \cos(2t) dt \\ &= -8 \left[ \frac{t}{2} - \frac{1}{4} \sin(2t) \right]_0^{2\pi} \\ &= -8 \left( \pi - 0 - \frac{1}{4} \sin(4\pi) + \frac{1}{4} \sin(0) \right) \\ &= -8\pi \end{aligned}$$

## 2. DO DIRECTLY

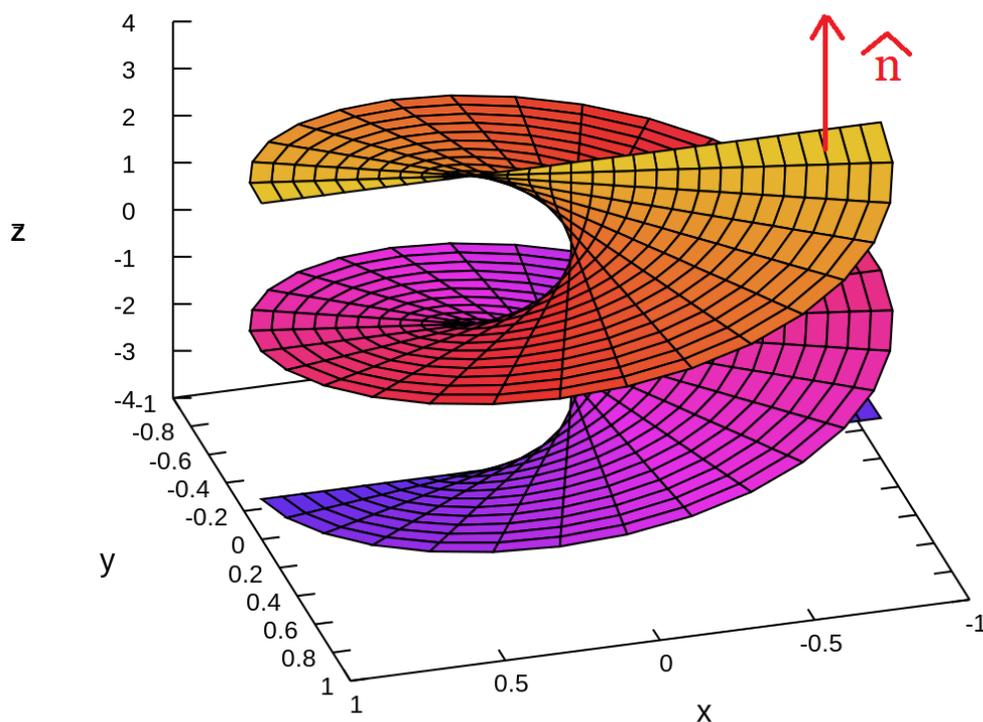
### Example 2:

$$\iint_S F \cdot d\mathbf{S}$$

$$F = \langle x, y, z^2 \rangle$$

$S$  : Helicoid parametrized by  $r(u, v) = \langle u \cos(v), u \sin(v), v \rangle$  with  $0 \leq u \leq 2$  and  $0 \leq v \leq 4\pi$

(1) **Picture:**



(2) **Parametrize  $S$ :**

$$r(u, v) = \langle u \cos(v), u \sin(v), v \rangle$$

(3) **Normal Vector:**

$$r_u = \langle \cos(v), \sin(v), 0 \rangle$$

$$r_v = \langle -u \sin(v), u \cos(v), 1 \rangle$$

$$\begin{aligned}
 \hat{n} &= r_u \times r_v \\
 &= \begin{vmatrix} i & j & k \\ \cos(v) & \sin(v) & 0 \\ -u \sin(v) & u \cos(v) & 1 \end{vmatrix} \\
 &= \langle \sin(v), -\cos(v), u \cos^2(v) + u \sin^2(v) \rangle \\
 &= \left\langle \sin(v), -\cos(v), \underbrace{u}_{\geq 0} \right\rangle
 \end{aligned}$$

(4)

$$\begin{aligned}
 \int \int_S F \cdot d\mathbf{S} &= \int \int_D F \cdot \hat{n} du dv \\
 &= \int \int_D \underbrace{\langle u \cos(v), u \sin(v), v^2 \rangle}_{\langle x, y, z^2 \rangle} \cdot \underbrace{\langle \sin(v), -\cos(v), u \rangle}_{\hat{n}} du dv \\
 &= \int_0^{4\pi} \int_0^2 \cancel{u \cos(v) \sin(v)} - \cancel{u \sin(v) \cos(v)} + uv^2 du dv \\
 &= \left( \int_0^2 u du \right) \left( \int_0^{4\pi} v^2 dv \right) \\
 &= \frac{128\pi^3}{3}
 \end{aligned}$$

### 3. DIVERGENCE THEOREM

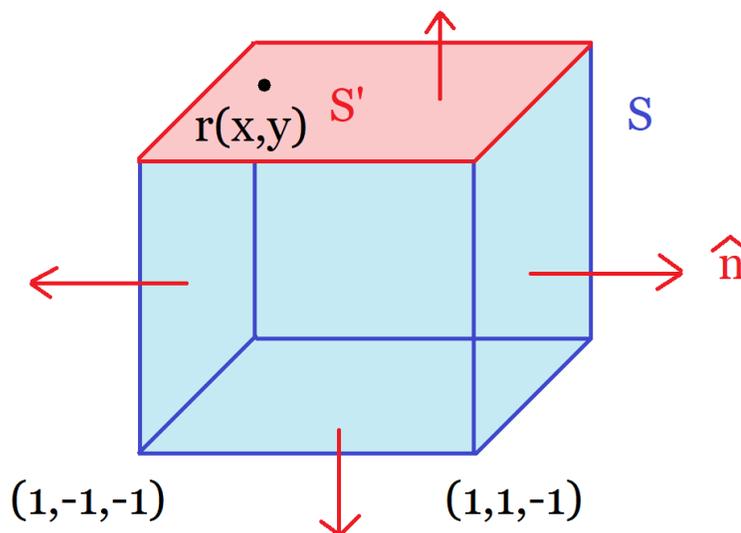
**Example 3:**

$$\int \int_S F \cdot d\mathbf{S}$$

$$F = \langle xy^2, x^2z, e^y \rangle$$

$S$  : Cube with vertices  $(\pm 1, \pm 1, \pm 1)$  oriented outwards, without the top

(1) **Picture:**



(2)

**Warning**

The Divergence Theorem only holds for closed surfaces!

Let  $S' =$  top of cube, then  $S + S'$  is closed, so by the divergence theorem

$$\begin{aligned}
\iint_{S+S'} F \cdot d\mathbf{S} &= \iiint_E \operatorname{div}(F) dx dy dz \\
&= \iiint_E (xy^2)_x + (x^2z)_y + (e^y)_z dx dy dz \\
&= \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 y^2 dx dy dz \\
&= (2)(2) \left[ \frac{y^3}{3} \right]_{-1}^1 \\
&= \frac{8}{3}
\end{aligned}$$

(3)

$$\begin{aligned}
\underbrace{\iint_{S+S'} F \cdot d\mathbf{S}}_{\frac{8}{3}} &= \underbrace{\iint_S F \cdot d\mathbf{S}}_{WTF} + \iint_{S'} F \cdot d\mathbf{S} \\
\iint_S F \cdot d\mathbf{S} &= \frac{8}{3} - \iint_{S'} F \cdot d\mathbf{S}
\end{aligned}$$

(4)  $\iint_{S'} F d\mathbf{S}$ **Parametrize  $S'$ :**  $r(x, y) = \langle x, y, 1 \rangle$ 

$$r_x = \langle 1, 0, 0 \rangle$$

$$r_y = \langle 0, 1, 0 \rangle$$

$$\begin{aligned}\hat{n} &= r_x \times r_y \\ &= \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} \\ &= \left\langle 0, 0, \underbrace{1}_{\geq 0} \right\rangle\end{aligned}$$

$$\begin{aligned}\int \int_{S'} F \cdot d\mathbf{S} &= \underbrace{xy, x^2(1), e^y}_{\langle xy^2, xz, e^y \rangle} \cdot \underbrace{\langle 0, 0, 1 \rangle}_{\hat{n}} dx dy \\ &= \int_{-1}^1 \int_{-1}^1 e^y dx dy \\ &= 2 \int_{-1}^1 e^y dy \\ &= 2(e - e^{-1})\end{aligned}$$

(5) **Answer:**

$$\int \int_S F \cdot d\mathbf{S} = \frac{8}{3} - 2(e - e^{-1})$$