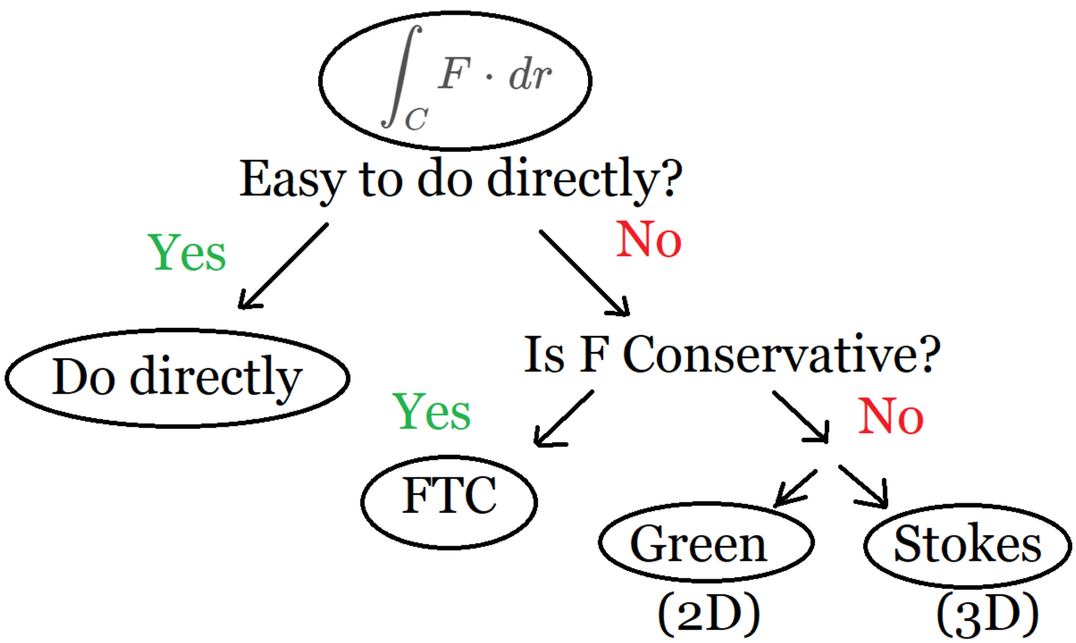
# LECTURE 27: FINAL EXAM REVIEW SESSION (II) - LINE INTEGRALS

There's a saying in German that says: "Everything has an end, except for a sausage, which has two!" And with this I would like to welcome you to the last (unofficial) lecture of Math 2E and the second part of the final exam review session!

**Today:** Is all about what to do when you face a random line integral! And like the last lecture, there is a nice roadmap for that:




*Date*: Friday, March 13, 2020.

# 1. Do Directly

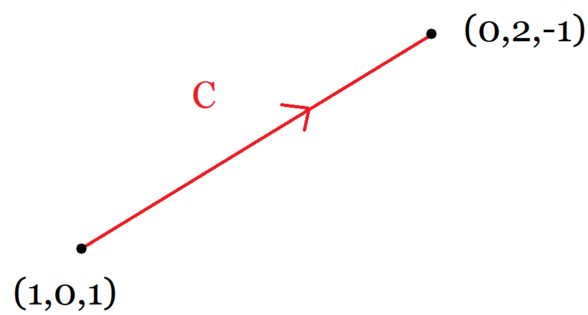
**Example 1:**

$\int_C F \cdot dr$

$F = \left\langle x^2, 3y, 3xz \right\rangle$

$C$ : Line connecting $(1,0,1)$ and $(0,2,-1)$

(1) **Picture:**




(2) **Parametrize $C$:**

$$\begin{cases} x(t) = (1-t)1 + t(0) = 1-t \\ y(t) = (1-t)0 + t(2) = 2t \\ z(t) = (1-t)1 + t(-1) = 1-2t \end{cases}$$

$(0 \leq t \leq 1)$

$r(t) = \langle 1-t, 2t, 1-2t \rangle$

(3)

$$\begin{aligned}
\int_C F \cdot dr &= \int_0^1 F(r(t)) \cdot r'(t)dt \\
&= \int_0^1 \underbrace{\left\langle (1-t)^2, 3(2t), 3(1-t)(1-2t) \right\rangle}_{\langle x^2, 3y, 3xz \rangle} \cdot \langle -1, 2, -2 \rangle \, dt \\
&= \int_0^1 -(1-t)^2 + 12t - 6(1-t)(1-2t)dt \\
&= \cdots \quad \text{(Expand out)} \\
&= \frac{14}{3}
\end{aligned}$$

## 2. FTC for Line Integrals

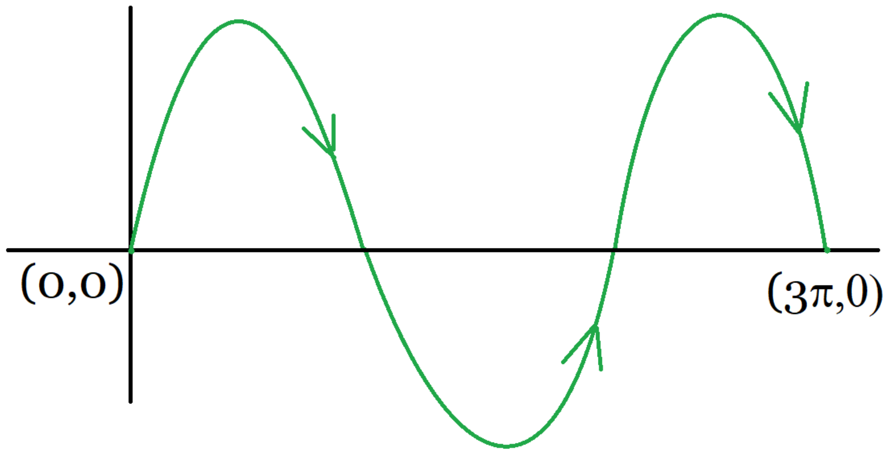
**Example 2:**

$\int_C F \cdot dr$

$F = \left\langle xy^2, x^2y \right\rangle$

$C$ : Arc of $y = \sin(x)$ from $(0,0)$ to $(3\pi, 0)$

(1) **Picture:**




(2) $F$ **conservative?**

(This is **NOT** a waste of time, have to do in any case!)

$$\text{Check: } Q_x - P_y = \left(x^2y\right)_x - \left(xy^2\right)_y = 2xy - 2xy = 0\checkmark$$

(3) **Find** $f$

$$F = \nabla f \Rightarrow \left\langle xy^2, x^2y \right\rangle = \langle f_x, f_y \rangle$$

$$\begin{aligned} f_x = xy^2 &\Rightarrow f(x,y) = \frac{1}{2}x^2y^2 + \text{ JUNK} \\ f_y = x^2y &\Rightarrow f(x,y) = \frac{1}{2}x^2y^2 + \text{ JUNK} \end{aligned}$$

Hence $f(x,y) = \frac{1}{2}x^2y^2$

(4)

$$\int_C F \cdot dr = f(3\pi, 0) - f(0,0) = \frac{1}{2}(3\pi)^2 0^2 - \frac{1}{2}0^2 0^2 = 0$$

**Note:** In 3 dimensions, you need to check $\text{curl}(F) = \langle 0, 0, 0 \rangle$ (just like on the mock exam)

## 3. Green's Theorem

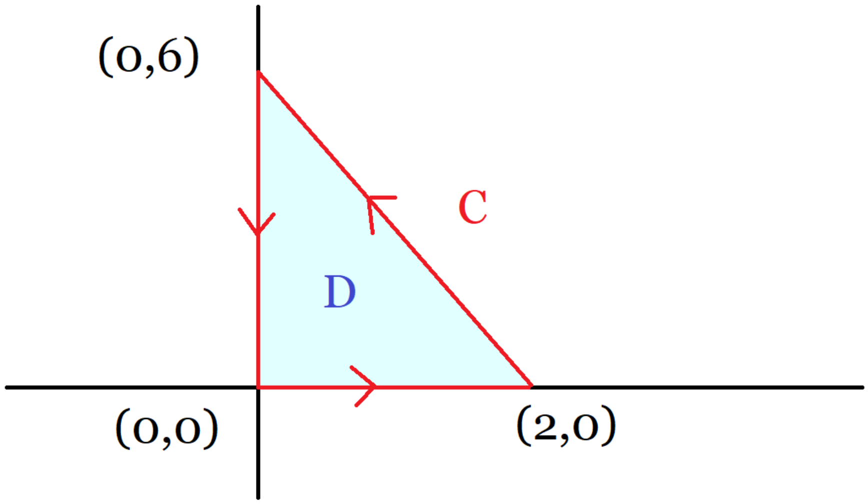
**Example 3:**

$\int_C F \cdot dr$

$F = \left\langle x^2 y, xy^2 \right\rangle$

$C$ : Triangle with vertices $(0,0), (2,0), (0,6)$ (counterclockwise)

(1) **Picture:**




(Too painful to do it directly)

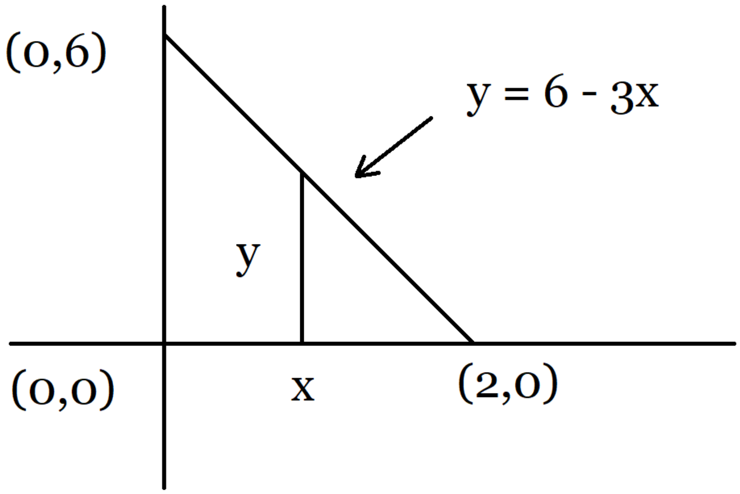
(2) $F$ **conservative?**

$$Q_x - P_y = \left(xy^2\right)_x - \left(x^2y\right)_y = y^2 - x^2$$

**NOT** Conservative (and 2 dimensions) $\Rightarrow$ Green!

(3)

$$\begin{aligned}
\int_C F \cdot dr &= \int\int_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dxdy \\
&= \int\int_D y^2 - x^2 dxdy \\
&= \int_0^2 \int_0^{6-3x} y^2 - x^2 dxdy \\
&= \cdots \\
&= 32
\end{aligned}$$




## 4. Stokes' Theorem

**Example 4:**

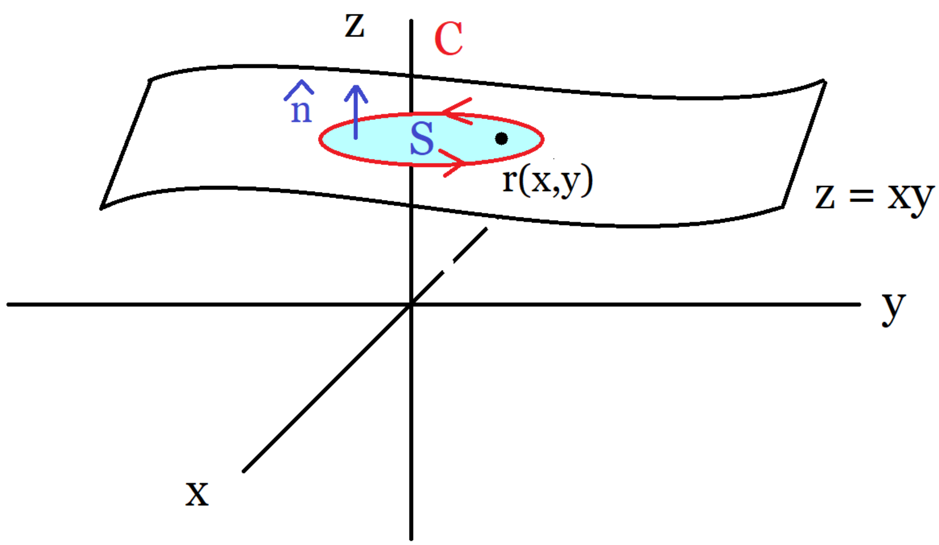
$\int_C F \cdot dr$

$F = \langle \sin(x), \cos(y), xz \rangle$

$C$ : Curve parametrized by $r(t) = \langle \cos(t), \sin(t), \cos(t)\sin(t) \rangle$ with $0 \leq t \leq 2\pi$

**Hint:** $C$ lies on the surface $z = xy$

(1) **Picture:**




(2) $F$ **conservative?**

$$\begin{aligned}\operatorname{curl}(F) &= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin(x) & \cos(y) & xz \end{vmatrix} \\ &= \left\langle \frac{\partial}{\partial y}(xz) - \frac{\partial}{\partial z}(\cos(y)), -\frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial z}(\sin(x)) \right. \\ &\quad \left. , \frac{\partial}{\partial x}(\cos(y)) - \frac{\partial}{\partial y}(\sin(x)) \right\rangle \\ &= \langle 0, -z, 0 \rangle \\ &\neq \langle 0, 0, 0 \rangle \end{aligned}$$

(3) **Stokes' Theorem**

$$\int_C F \cdot dr = \int\int_S \operatorname{curl}(F) \cdot d\mathbf{S}$$

(4) **Parametrize $S$:**

$$r(x,y) = \langle x, y, xy \rangle$$

$$\begin{aligned} r_x &= \langle 1, 0, y \rangle \\ r_y &= \langle 0, 1, x \rangle \end{aligned}$$

$$\hat{n} = r_x \times r_y = \begin{vmatrix} i & j & k \\ 1 & 0 & y \\ 0 & 1 & x \end{vmatrix} = \left\langle -y, -x, \underbrace{1}_{\geq 0} \right\rangle \checkmark$$

(5)

$$\begin{aligned} \int\int_S \operatorname{curl}(F) \cdot d\mathbf{S} &= \int\int_D \underbrace{\langle 0, -xy, 0 \rangle}_{\langle 0, -z, 0 \rangle} \cdot \underbrace{\langle -y, -x, 1 \rangle}_{\hat{n}} dxdy \\ &= \int\int_D x^2 y dxdy \qquad \text{D: Disk of Radius 1} \\ &= \int_0^{2\pi} \int_0^1 r^2 \cos^2(\theta) r \sin(\theta) r dr d\theta \\ &= \left( \int_0^1 r^4 dr \right) \left( \int_0^{2\pi} \cos^2(\theta) \sin(\theta) d\theta \right) \\ &= \frac{1}{5} \left[ -\frac{\cos^3(\theta)}{3} \right]_0^{2\pi} \\ &= 0 \end{aligned}$$