LECTURE 28: FINAL EXAM REVIEW SESSION (III) - MMMMMMH, DONUTS!

Welcome to a special encore episode of *That's so Peyam!* Today we'll talk about all the non-vector field topics (like surface integrals of functions, or Jacobians), but this time in the context of... donuts!

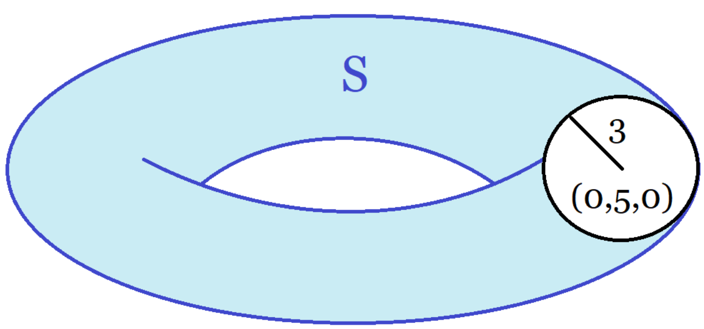
1. Parametrization

Here is the main example that we'll talk about today:

Example 1:

Let S be the donut obtained by rotating the circle centered at $(0, 5, 0)$ and radius 3 about the $z-$axis.

Picture:

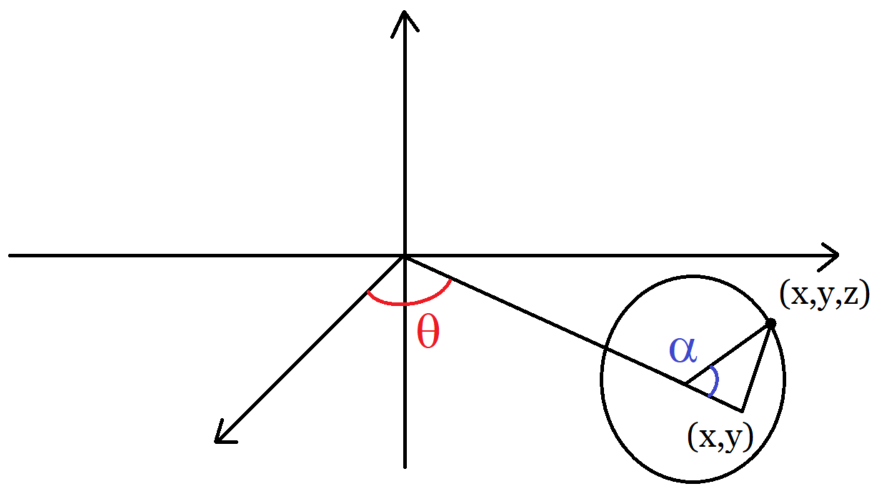


Date: Friday, March 13, 2020.

Parametric Equations: (you don't need to know how to derive this)

$$\begin{cases} x = (5 + 3\cos(\alpha))\cos(\theta) \\ y = (5 + 3\cos(\alpha))\sin(\theta) \\ z = 3\sin(\alpha) \end{cases}$$

$0 \le \theta \le 2\pi, 0 \le \alpha \le 2\pi$



$\theta =$ angle between x-axis and $(x, y)\alpha =$ angle between (x, y) and (x, y, z)

2. Surface Area

Video: Surface Area of a Donut

Example 2:

Find the surface area of S

Surface Area:

$$\text{Area } (S) = \int\int_S dS = \int\int_D \|r_\theta \times r_\alpha\| \, d\theta d\alpha$$

(It's basically $dS = \|r_u \times r_v\| \, dudv$ but with θ and α instead)

(1)

$$r(\theta, \alpha) = \langle (5 + 3\cos(\alpha))\cos(\theta), (5 + 3\cos(\alpha))\sin(\theta), 3\sin(\alpha) \rangle$$

(2)

$$\begin{aligned} r_\theta &= \langle -(5 + 3\cos(\alpha))\sin(\theta), (5 + 3\cos(\alpha))\cos(\theta), 0 \rangle \\ r_\alpha &= \langle -3\sin(\alpha)\cos(\theta), -3\sin(\alpha)\sin(\theta), 3\cos(\alpha) \rangle \end{aligned}$$

(3)

$$\begin{aligned} r_\theta \times r_\alpha &= \begin{vmatrix} i & j & k \\ & r_\theta & \\ & r_\alpha & \end{vmatrix} \\ &= \text{A Nightmare} \\ &= \underbrace{3(5 + 3\cos(\alpha))}_{\geq 0} \langle \cos(\alpha)\cos(\theta), \cos(\alpha)\sin(\theta), -\sin(\alpha) \rangle \end{aligned}$$

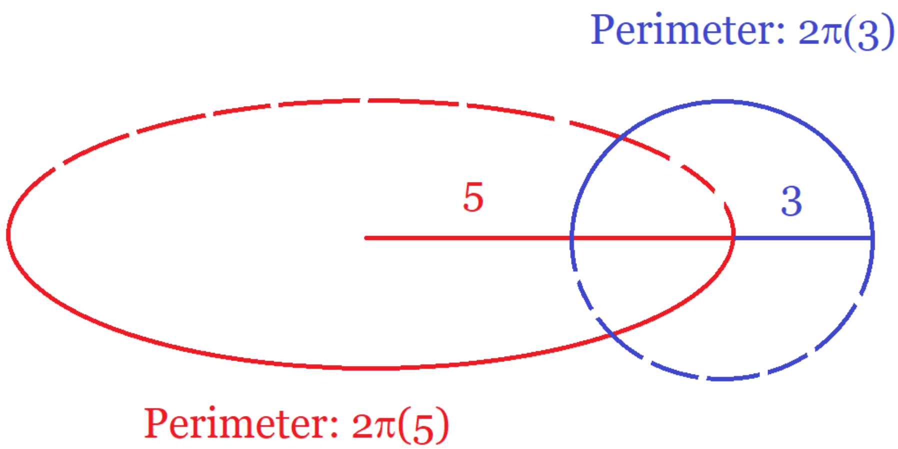
(4) dS

$$
\begin{aligned}
dS =& \|r_\theta \times r_\alpha\| \\
=& 3(5+3\cos(\alpha))\,\|\langle \cos(\alpha)\cos(\theta), \cos(\alpha)\sin(\theta), -\sin(\alpha)\rangle\| \\
=& 3(5+3\cos(\alpha))\sqrt{\underbrace{\cos^2(\alpha)\cos^2(\theta)+\cos^2(\alpha)\sin^2(\theta)}_{\cos^2(\alpha)}+\sin^2(\theta)} \\
=& 3(5+3\cos(\alpha))\sqrt{\underbrace{\cos^2(\alpha)+\sin^2(\alpha)}_{1}} \\
=& 3(5+3\cos(\alpha)) \\
=& 15+9\cos(\alpha)
\end{aligned}
$$

(5)

$$
\begin{aligned}
\text{Area } =& \int_0^{2\pi}\int_0^{2\pi} 15+9\cos(\alpha)d\theta d\alpha \\
=& (15)(2\pi)(2\pi) \\
=& 60\pi^2
\end{aligned}
$$

Remark: $15(2\pi)(2\pi) = (3)(5)(2\pi)(2\pi) = (2\pi 3)(2\pi 5)$. This makes sense, because, in order to get the donut, you're revolving a circle of radius 3 (with perimeter $2\pi 3$) around a circle of radius 5 (with perimeter $2\pi 5$)



(So in a sense, a donut is a circle times a circle)

3. Surface Integrals of Functions

Example 3:

Calculate $\int\int_S zdS$

$$\begin{aligned}\int\int_S zdS &= \int\int_D f(r(\theta,\alpha))\left\lVert r_\theta \times r_\alpha\right\rVert d\theta d\alpha \\ &= \int_0^{2\pi}\int_0^{2\pi} \underbrace{3\sin(\alpha)}_{z}\underbrace{(15+9\cos(\alpha))d\theta d\alpha}_{dS} \\ &= \cdots \\ &= 0\end{aligned}$$

4. Change of Variables

Last but not least, for our Math 2E grand finale, let's:

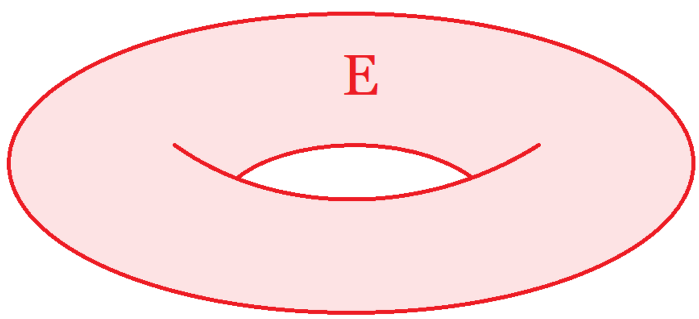
Example 4:

Calculate the volume of the donut!

(Actually a hidden Jacobian problem!)

(1)

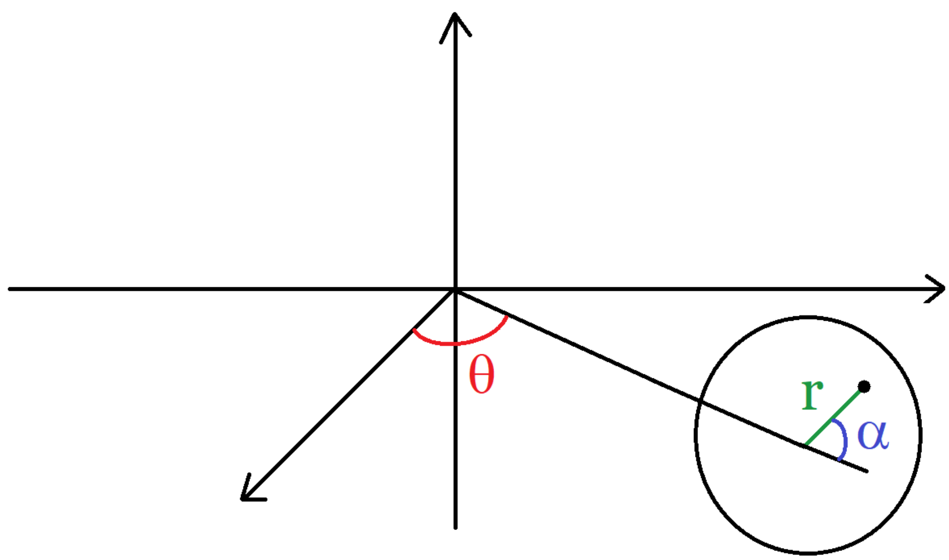
$$V = \int\int\int_E 1 dx dy dz$$



(2) **Donut Coordinates** (will be given)

$$\begin{cases} x = (5 + r\cos(\alpha))\cos(\theta) \\ y = (5 + r\sin(\alpha))\cos(\theta) \\ z = r\sin(\alpha) \end{cases}$$

Here θ and α are as before, and r is the radius:



(3) **Find** E'

$$0 \leq r \leq 3$$
$$0 \leq \theta \leq 2\pi$$
$$0 \leq \alpha \leq 2\pi$$

(This change of variables takes a donut and turns it into a box, wow!)

(4) **Jacobian:**

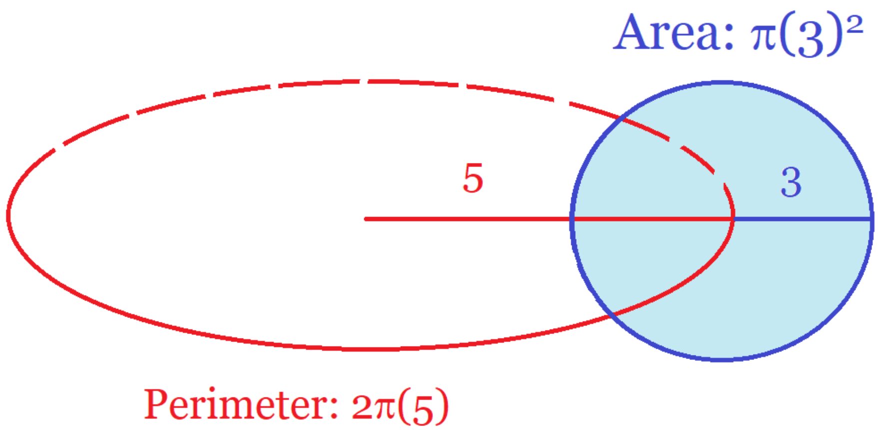
$$dxdydz = \left|\frac{dxdydz}{drd\theta d\alpha}\right| drd\theta d\alpha$$

$$\begin{aligned}\frac{dxdydz}{drd\theta d\alpha} &= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \alpha} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \alpha} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \alpha} \end{vmatrix} \\ &= \text{Some Nightmare} \\ &= r(5 + r\cos(\alpha))\end{aligned}$$

(5)

$$\begin{aligned}V &= \int\int\int_E 1 dxdydz \\ &= \int\int\int_{E'} 1r(3 + r\cos(\alpha))drd\theta d\alpha \\ &= \int_0^{2\pi}\int_0^{2\pi}\int_0^3 3r + r^2\cos(\alpha)drd\theta d\alpha \\ &= (2\pi)(2\pi)\left[\frac{5}{2}r^2\right]_0^3 \\ &= (2\pi)(2\pi)\frac{5}{2}3^2 \\ &= 90\pi^2\end{aligned}$$

Note: Same as $(2\pi 5)(\pi 3^2)$
(So in a sense, the full donut is a disk times a circle)



And with this, I would like to officially thank you for flying Peyam Airlines; it's been a pleasure having you on board, and I wish you a safe onward journey!

The End