LECTURE 28: FINAL EXAM REVIEW SESSION (III) - MMMMMH, DONUTS!

Welcome to a special encore episode of That’s so Peyam! Today we’ll talk about all the non-vector field topics (like surface integrals of functions, or Jacobians), but this time in the context of... donuts!

1. Parametrization

Here is the main example that we’ll talk about today:

**Example 1:**

Let $S$ be the donut obtained by rotating the circle centered at $(0, 5, 0)$ and radius 3 about the $z$-axis.

**Picture:**

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*Date: Friday, March 13, 2020.*
Parametric Equations: (you don’t need to know how to derive this)

\[
\begin{align*}
    x &= (5 + 3 \cos(\alpha)) \cos(\theta) \\
    y &= (5 + 3 \cos(\alpha)) \sin(\theta) \\
    z &= 3 \sin(\alpha)
\end{align*}
\]

\(0 \leq \theta \leq 2\pi, 0 \leq \alpha \leq 2\pi\)

\(\theta = \text{angle between x-axis and } (x, y)\) \\
\(\alpha = \text{angle between } (x, y) \text{ and } (x, y, z)\)

2. Surface Area

Video: [Surface Area of a Donut](#)

Example 2:

Find the surface area of \(S\)
Surface Area:

\[
\text{Area } (S) = \int \int_S dS = \int \int_D \| r_\theta \times r_\alpha \| \, d\theta d\alpha
\]

(It's basically \( dS = \| r_u \times r_v \| \, du \, dv \) but with \( \theta \) and \( \alpha \) instead)

(1)

\[ r(\theta, \alpha) = \langle (5 + 3 \cos(\alpha)) \cos(\theta), (5 + 3 \cos(\alpha)) \sin(\theta), 3 \sin(\alpha) \rangle \]

(2)

\[
\begin{align*}
  r_\theta &= \langle -(5 + 3 \cos(\alpha)) \sin(\theta), (5 + 3 \cos(\alpha)) \cos(\theta), 0 \rangle \\
  r_\alpha &= \langle -3 \sin(\alpha) \cos(\theta), -3 \sin(\alpha) \sin(\theta), 3 \cos(\alpha) \rangle
\end{align*}
\]

(3)

\[
\begin{vmatrix}
  i & j & k \\
  r_\theta & r_\alpha \end{vmatrix}
= \text{A Nightmare}
= 3(5 + 3 \cos(\alpha)) \langle \cos(\alpha) \cos(\theta), \cos(\alpha) \sin(\theta), -\sin(\alpha) \rangle
\geq 0
\]

(4) \( dS \)
\[ dS = \| r_\theta \times r_\alpha \| \]
\[ = 3(5 + 3 \cos(\alpha)) \| (\cos(\alpha) \cos(\theta), \cos(\alpha) \sin(\theta), -\sin(\alpha)) \| \]
\[ = 3(5 + 3 \cos(\alpha)) \sqrt{\cos^2(\alpha) \cos^2(\theta) + \cos^2(\alpha) \sin^2(\theta) + \sin^2(\theta)} \]
\[ = 3(5 + 3 \cos(\alpha)) \sqrt{\cos^2(\alpha) + \sin^2(\alpha)} \]
\[ = 3(5 + 3 \cos(\alpha)) \]
\[ = 15 + 9 \cos(\alpha) \]

(5)

\[ \text{Area} = \int_0^{2\pi} \int_0^{2\pi} 15 + 9 \cos(\alpha) d\theta d\alpha \]
\[ = (15)(2\pi)(2\pi) \]
\[ = 60\pi^2 \]

**Remark:** \( 15(2\pi)(2\pi) = (3)(5)(2\pi)(2\pi) = (2\pi 3)(2\pi 5) \). This makes sense, because, in order to get the donut, you’re revolving a circle of radius 3 (with perimeter \( 2\pi 3 \)) around a circle of radius 5 (with perimeter \( 2\pi 5 \)).
3. Surface Integrals of Functions

Example 3:

Calculate $\int \int_S zdS$

\[
\int \int_S zdS = \int \int_D f(r(\theta, \alpha)) \| r_\theta \times r_\alpha \| d\theta d\alpha \\
= \int_0^{2\pi} \int_0^{2\pi} 3 \sin(\alpha) (15 + 9 \cos(\alpha)) d\theta d\alpha \\
= \cdots \\
= 0
\]
4. **Change of Variables**

Last but not least, for our Math 2E grand finale, let’s:

**Example 4:**

Calculate the volume of the donut!

(Actually a hidden Jacobian problem!)

\[ V = \iiint_E 1 \, dxdydz \]

(2) **Donut Coordinates** (will be given)
Here $\theta$ and $\alpha$ are as before, and $r$ is the radius:

\[
\begin{align*}
    x &= (5 + r \cos(\alpha)) \cos(\theta) \\
    y &= (5 + r \sin(\alpha)) \cos(\theta) \\
    z &= r \sin(\alpha)
\end{align*}
\]

(3) Find $E'$

\[
0 \leq r \leq 3 \\
0 \leq \theta \leq 2\pi \\
0 \leq \alpha \leq 2\pi
\]

(This change of variables takes a donut and turns it into a box, wow!)
(4) **Jacobian:**

\[
dx\,d\gamma\,dz = \begin{vmatrix} \frac{dx}{dr} & \frac{dx}{\theta} & \frac{dx}{\alpha} \\ \frac{d\gamma}{dr} & \frac{d\gamma}{\theta} & \frac{d\gamma}{\alpha} \\ \frac{dz}{dr} & \frac{dz}{\theta} & \frac{dz}{\alpha} \end{vmatrix} \, dr\,d\theta\,d\alpha
\]

\[
dx\,d\gamma\,dz \over dr\,d\theta\,d\alpha = \begin{vmatrix} \partial_x & \partial_x & \partial_x \\ \partial_\gamma & \partial_\gamma & \partial_\gamma \\ \partial_z & \partial_z & \partial_z \end{vmatrix} = \text{Some Nightmare}
\]

\[= r(5 + r \cos(\alpha))\]

(5)

\[
V = \int \int \int_E 1\,dx\,d\gamma\,dz
\]

\[= \int \int \int_{E'} r(3 + r \cos(\alpha))\,dr\,d\theta\,d\alpha
\]

\[= \int_0^{2\pi} \int_0^{2\pi} \int_0^3 (5 + r^2 \cos(\alpha))dr\,d\theta\,d\alpha
\]

\[= (2\pi)(2\pi) \left[ \frac{5 r^2}{2} \right]_0^3
\]

\[= (2\pi)(2\pi) \frac{5}{2} 3^2
\]

\[= 90\pi^2
\]

**Note:** Same as \((2\pi5)(\pi3^2)\)

(So in a sense, the full donut is a disk times a circle)
And with this, I would like to officially thank you for flying Peyam Airlines; it’s been a pleasure having you on board, and I wish you a safe onward journey!

The End