LECTURE 2: REVIEW OF PREREQUISITES

Readings:

- Section 1 of the lecture notes on Convolution
- Section 2 of the lecture notes on the Dominated Convergence Theorem
- Section 3 of the lecture notes on the Polar Coordinates Formula
- Section 4 of the lecture notes on Integration by Parts
- Section 2.2.1b: Poisson's Equation (pages 22-25)

1. Convolution

1.1. **Definition.** Most of this week is focused on review of some of the 'prerequisites' needed to survive the rest of the readings.

First of all, we need a *smart* way of multiplying two functions called **convolution**, which is quintessential in analysis:

Convolution:

$$(f \star g)(x) = \int_{\mathbb{R}^n} f(y)g(x-y)dy = \int_{\mathbb{R}^n} f(x-y)g(y)dy$$

Date: Monday, April 6, 2020.

(Always think: The sum has to be x, so $y+(x-y) = x$ and $x-y+y = x$).

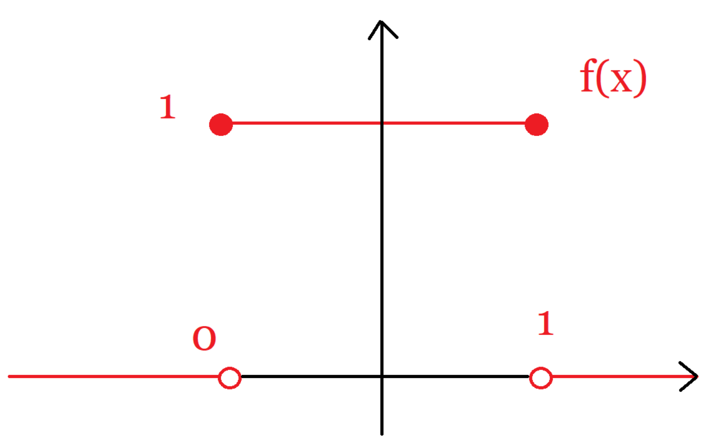
To emphasize that this is a function of x, let's do the following:

Example:

$(n = 1)$ Calculate $f \star g$, where

$$f(x) = 1_{[0,1]}(x) = \begin{cases} 1 \text{ if } & x \in [0,1] \\ 0 \text{ if } & x \notin [0,1] \end{cases}$$

$$g(x) = e^x$$



$$\begin{aligned}
(f \star g)(x) &= \int_{\mathbb{R}} f(y)g(x-y)dy \\
&= \int_{-\infty}^{\infty} 1_{[0,1]}(y)e^{x-y}dy \\
&= \int_{0}^{1} 1e^{x-y}dy \qquad \text{Since } f = 0 \text{ outside } [0,1] \\
&= e^x \int_0^1 e^{-y}dy \\
&= e^x(1-e^{-1})
\end{aligned}$$

Which indeed is a function of x only.

Note: Here you can find a beautiful demo about what the convolution of two functions looks like: Convolution Demo. You can also change your functions at the top if you like

1.2. **Some Intuition:**

Video: Convolution Intuition

Convolution is just a fancy way of multiplying two functions, in the following sense:

Example: What is the coefficient of x^2 in:

$$\left(x^2+2x+3\right)\left(2x^2+4x+1\right) = (1 \times 1 + 2 \times 4 + 3 \times 2)\, x^2 + \cdots = 15x^2 + \cdots$$

Notice: To find the coefficient of x^2 you're essentially looking at all the terms whose exponents sum to 2.

So the coefficient of x^2 in $\left(a_2x^2 + a_1x + a_0\right)\left(b_2x^2 + b_1x + b_0\right)$ is

$$c_2 = \sum_{i+j=2} a_ib_j = a_0b_2 + a_1b_1 + a_2b_0 = \sum_{i=0}^{2} a_ib_{2-j}$$

Fact:

In general, the coefficient of x^k in

$(a_nx^n + \cdots + a_0)\,(b_nx^n + \cdots + a_0)$ is

$$c_k = \sum_{i+j=k} a_ib_j = \sum_{i=0}^{k} a_ib_{k-i}$$

Compare this to:

$$(f \star g)(x) = \int_{\mathbb{R}^n} f(y)g(x-y)dy$$

In other words, convolution is just the continuous analog of the above fact! In other words, $(f \star g)(x)$ is the x-th coefficient of f times g (where here times is like a polynomial multiplication)

1.3. **Relationship with PDEs.** Recall that $\Phi(x) = \frac{C}{|x|^{n-2}}$ (fundamental solution), solves $-\Delta\Phi = 0$. The question is now: How do you solve $-\Delta u = f$ for any f? It turns out that the following is true:

Theorem:

Let $u(x) = (\Phi \star f)(x)$, then $-\Delta u = f$

This is a very beautiful but nontrivial result, which requires delicate analysis (see section 2.2.1b)

2. The Dominated Convergence Theorem

Motivation: Very often in this course, we will need to pass to the limit inside an integral

Question:

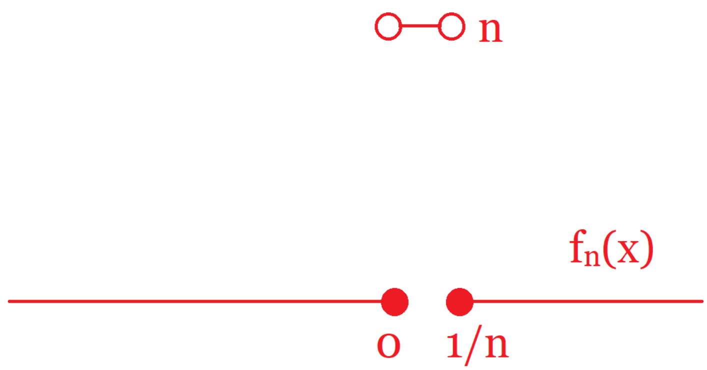
Suppose $f_n \to f$ pointwise (that is for every x, $f_n(x) \to f(x)$ as n goes to ∞), does it follow that $\int_{\mathbb{R}^n} f_n(x)dx \to \int_{\mathbb{R}^n} f(x)dx$?

In other words, does $\lim_{n\to\infty} \int f_n = \int \lim_{n\to\infty} f_n$?

In general, the answer is **NO**:

Example:

$$f_n(x) = n1_{\left(0,\frac{1}{n}\right)} = \begin{cases} n \text{ if } \ x \in \left(0, \frac{1}{n}\right) \\ 0 \text{ otherwise} \end{cases}$$



Claim: $f_n \to f(x) =: 0$ pointwise

Why? If $x \notin (0,1)$, then $f_n(x) = 0 \to 0$ as $n \to \infty$.

And if $x \in (0,1)$, then for n_0 large enough, $\frac{1}{n_0} < x$ and so if $n \geq n_0$, $f_n(x) = 0 \to 0$ as $n \to \infty$.

Claim: $\int f_n \not\to \int f$

$$\int_{-\infty}^{\infty} f_n(x)dx = \int_0^{\frac{1}{n}} n = n\left(\frac{1}{n} - 0\right) = 1$$

$$1 = \int_{-\infty}^{\infty} f_n \not\to \int_{-\infty}^{\infty} f = \int_{-\infty}^{\infty} 0 = 0$$

2.1. **The Theorem:**

Question:

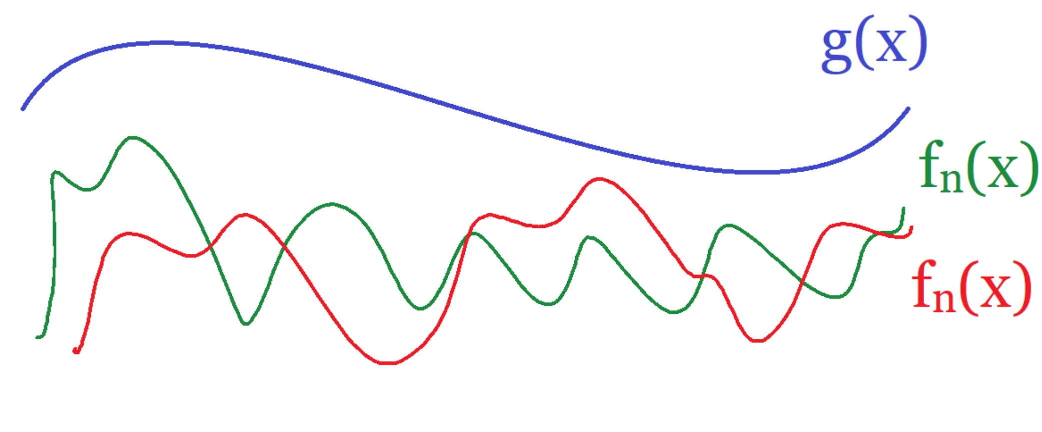
Under what conditions do we have $\lim_{n\to\infty} \int f_n = \int f$

It turns out that there is a very elegant condition: If somehow we can control the f_n in a way that's independent of n, then we're good! And this is the essence of:

Dominated Convergence Theorem:

Suppose $f_n \to f$ pointwise and suppose $|f_n(x)| \leq g(x)$ for some g independent of n with $\int g(x) < \infty$, then

$$\int f_n \to \int f$$



In particular, no matter how wild the f_n are, as long as they're trapped inside a function g whose integral is finite, then in fact we can pass in the limit inside the integral

Example:

Suppose $\int |g(x)|\, dx < \infty$ and $|f'(x)| < C < \infty$ for some $C > 0$, then does

$$\int g(x)\left(\frac{f(x+h)-f(x)}{h}\right) dx \to \int g(x)f'(x)dx \text{ as } \ h \to 0?$$

(This is actually the first step in the solution to Poisson's equation later in 2.2.1b)

Let $f_h(x) = g(x)\left(\frac{f(x+h)-f(x)}{h}\right)$

Enough to show $|f_h| \leq \widetilde{g}$ where $\widetilde{g}$ is some function with $\int \widetilde{g} < \infty$

But by the mean value theorem from calculus, $\frac{f(x+h)-f(x)}{h} = f'(c)$ for some c, so

$$\left|\frac{f(x+h)-f(x)}{h}\right| = |f'(c)| < C < \infty$$

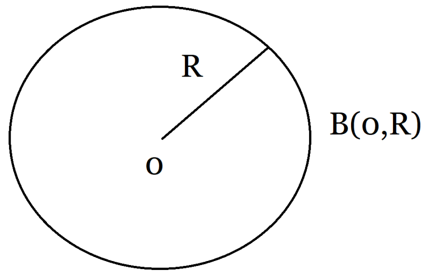
And so:

$$\left|g(x)\left(\frac{f(x+h)-f(x)}{h}\right)\right| = |g(x)|\,|f'(c)| \leq \underbrace{C\,|g(x)|}_{\widetilde{g}(x)}$$

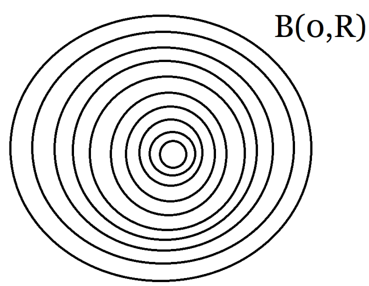
And moreover $\int \tilde{g}(x) = C\int |g(x)| < \infty$ (by assumption)
So the result follows from the dominated convergence theorem $\square$

Note: Just know that whenever we pass a limit inside the integral, we're implicitly doing the dominated convergence theorem process above!

3. The Polar Coordinates Formula



Very often in this course we'll need to integrate a function over a ball $B(0, R)$, which, in most cases, is hard to do. It turns out, however, that it's sometimes easier to decompose the ball into concentric spheres (think onion rings) and integrate over all the spheres

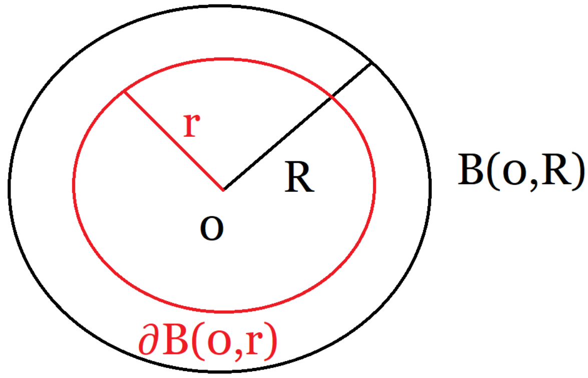


Note: This is especially useful if you have a function that depends on $|x|$

Polar Coordinates Formula:

$$\int_{B(0,R)} f(x)dx = \int_0^R \left(\int_{\partial B(0,r)} f(x)dS(x) \right) dr$$

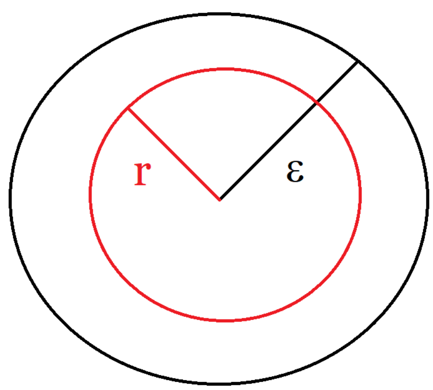
(Think like summing up the integrals on the spheres)



Example:

Calculate $\int_{B(0,\epsilon)} \Phi(x)dx$

Here again $\Phi(x) = \frac{C_n}{|x|^{n-2}}$ with $C_n = \frac{1}{n(n-2)\alpha(n)}$ and $\alpha(n) = |B(0,1)|$ (Volume)



By polar coordinates, we get:

$$
\begin{aligned}
\int_{B(0,\epsilon)} \Phi(x)dx &= \int_0^\epsilon \left(\int_{\partial B(0,r)} \Phi dS \right) dr \\
&= \int_0^\epsilon \left(\int_{\partial B(0,r)} \frac{C_n}{|x|^{n-2}} dS \right) dr \\
&\quad \text{(Remember } |x| = r \text{ on } \partial B(0,r)) \\
&= C_n \int_0^\epsilon \left(\int_{\partial B(0,r)} \frac{1}{r^{n-2}} dS \right) dr \\
&= C_n \int_0^\epsilon \frac{1}{r^{n-2}} \left(\int_{\partial B(0,r)} 1 dS \right) dr \\
&= C_n \int_0^\epsilon \frac{1}{r^{n-2}} \underbrace{|\partial B(0,r)|}_{\text{Surface Area}} dr
\end{aligned}
$$

Fact:

$$|B(0,r)| = \alpha(n) r^n$$

(Think for instance $\frac{4}{3}\pi r^3$ in $n = 3$)

$$|\partial B(0,r)| = n\alpha(n) r^{n-1}$$

(Think $4\pi r^2 = \left(\frac{4}{3}\pi r^3\right)'$)

Hence we get

$$
\begin{aligned}
\int_{B(0,\epsilon)} \Phi(x)dx =& C_n \int_0^\epsilon \frac{1}{r^{n-2}} \left|\partial B(0,r)\right| dr \\
=& C_n \int_0^\epsilon \frac{1}{r^{n-2}} n\alpha(n) r^{n-1} \\
=& C_n n\alpha(n) \int_0^\epsilon r dr \\
=& C_n n\alpha(n) \frac{\epsilon^2}{2} \\
=& \frac{1}{n(n-2)\alpha(n)} n\alpha(n) \frac{\epsilon^2}{2} \\
=& \frac{1}{2(n-2)} \epsilon^2 \\
=& O(\epsilon^2)
\end{aligned}
$$

4. Integration by Parts

The last thing that we'll need is a multidimensional version of integration by parts. In my opinion, it's the single most important formula in this course; we'll be using this over and over again

Goal:

Find a formula for $\int_U (\Delta u)v$

(In practice u is bad but v is good, so we'd like to put all the derivatives on v)

Note: The one-dimensional analog of $\int_U (\Delta u)\, v$ is

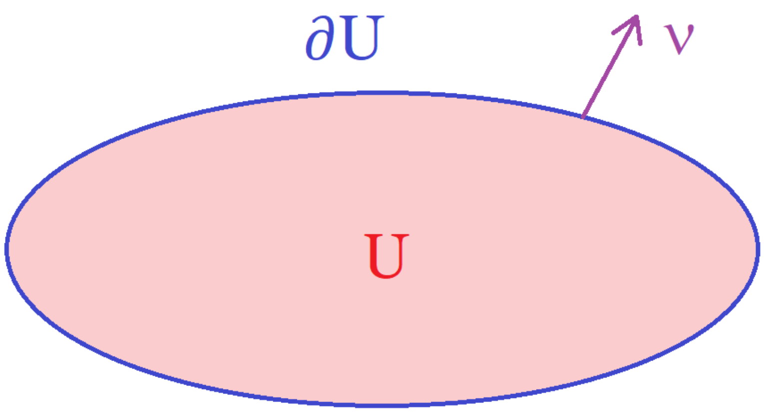
$$\int_a^b f''g = [f'g]_a^b - \int_a^b f'g'$$

So by analogy we should say

$$\int_U (\Delta u)\, v = \int_{\partial U} \cancel{Du} v dS - \int_U Du \cdot Dv$$

This definition is **almost correct**, except for two things: First of all, Du and Dv are vectors, so the correct way to multiply them is using the dot product, hence the reason why we wrote $Du \cdot Dv$.

More importantly, for the Duv term in ∂U, we would like to replace Du by a scalar that says something like “The derivative of u on ∂U. Now remember that U has a special feature called the normal vector ν, so somehow we would like to take the derivative of u in the direction of ν



Definition:

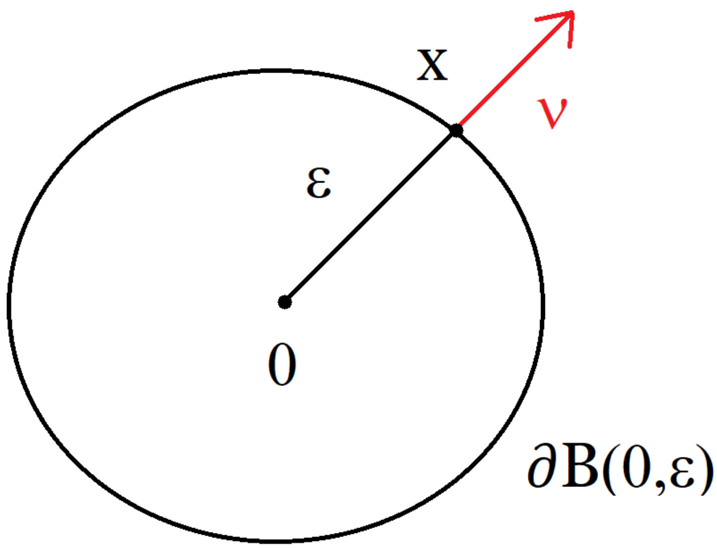
$$\frac{\partial u}{\partial \nu} = Du \cdot \nu$$

is called the **normal derivative of** u (= directional derivative in the direction of ν)

Interpretation: If $\frac{\partial u}{\partial \nu} > 0$, then u is flowing out of ∂U. And if $\frac{\partial u}{\partial \nu} = 0$, then u is stuck on ∂U (think like putting sticky glue on the boundary of U)

Example:

Calculate $\frac{\partial \Phi}{\partial \nu}$ on $\partial B(0, \epsilon)$



$$\frac{\partial \Phi}{\partial \nu} = D\Phi \cdot \nu$$

Find $D\Phi$:

$$\Phi(x) = \frac{C_n}{|x|^{n-2}} = C_n |x|^{2-n}$$

$$\Phi_{x_i} = C_n(2-n)|x|^{1-n}\frac{\partial |x|}{\partial x_i} = C_n(2-n)|x|^{1-n}\frac{x_i}{|x|} = C_n(2-n)|x|^{-n}x_i$$

$$\begin{aligned} D\Phi &= (\Phi_{x_1}, \cdots, \Phi_{x_n}) \\ &= \left(C_n(2-n)|x|^{-n}x_1, \cdots, C_n(2-n)|x|^{-n}x_n\right) \\ &= C_n(2-n)|x|^{-n}(x_1, \cdots, x_n) \\ &= C_n(2-n)|x|^{-n}x \end{aligned}$$

Find ν: Notice in the picture above that ν points in the same direction as x, so $\nu = Cx$ for some C, and to make ν have length 1, we get:

Fact:

$$\nu = \frac{x}{|x|}$$

IMPORTANT: This formula for ν is only true for the ball!!!

And so

$$
\begin{aligned}
\frac{\partial \Phi}{\partial \nu} =& D\Phi \cdot \nu \\
=& C_n(2-n)\left|x\right|^{-n} x \cdot \left(\frac{x}{|x|}\right) \\
=& C_n(2-n)\left|x\right|^{-n} \left(\frac{|x|^2}{|x|}\right) \\
=& C_n(2-n)\left|x\right|^{1-n} \\
=& C_n(2-n)\epsilon^{1-n} \quad \text{On } \partial B(0,\epsilon) \\
=& \frac{1}{n(n-2)\alpha(n)}(2-n)\epsilon^{1-n} \\
=& \frac{-1}{n\alpha(n)}\epsilon^{1-n}
\end{aligned}
$$

Since we now know the notion of a normal derivative, we can now state the correct integration by parts formula:

Integration by Parts:

$$
\int_U (\Delta u)\, v = \int_{\partial U} \left(\frac{\partial u}{\partial \nu}\right) v - \int_U Du \cdot Dv
$$

5. Poisson's Equation

Reading: Section 2.2.1b

Video: Poisson's Equation

Now that we know how to solve $-\Delta u = 0$, how can we solve $-\Delta u = f$ where f is a given function?

Main Idea: "Multiply" Φ with f, but this time where multiplication is convolution, that is

$$u = \Phi \star f = \int_{\mathbb{R}^n} \Phi(x-y) f(y) dy$$

Naively we get:

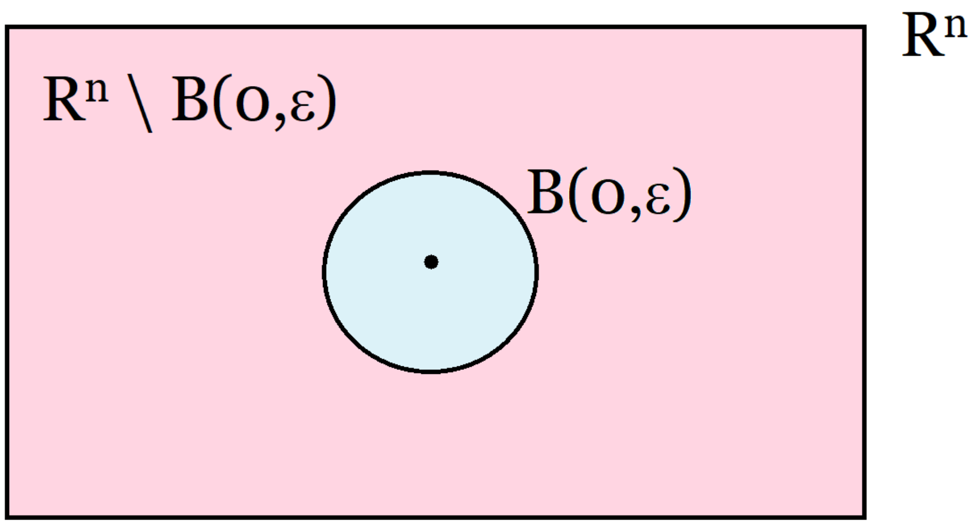
$$\Delta u = \Delta \left(\int_{\mathbb{R}^n} \Phi(x-y) f(y) dy \right) = \int_{\mathbb{R}^n} (\Delta \Phi) f = 0$$

But that is WRONG because Φ is very badly behaved at O.

Note: The following is handwavy and is just an overview (not a substitute) of the proof in the book. The main idea is to first put all the derivatives on f because f is much nicer than Φ

$$\Delta u = \Delta \left(\int_{\mathbb{R}^n} \Phi(x-y) f(y) dy \right) = \int_{\mathbb{R}^n} \Phi(y) \Delta f(x-y)$$

And then do some surgery, that is decomposing $\mathbb{R}^n$ as $B(0, \epsilon)$ (where Φ is ill-behaved) and $R^n \backslash B(0, \epsilon)$ (where Φ is well-behaved)



Then decompose the above integral as:

$$\Delta u = \int_{\mathbb{R}^n} \Phi \Delta f = \underbrace{\int_{B(0,\epsilon)} \Phi \Delta f}_{\text{Show small}} + \int_{\mathbb{R}^n \setminus B(0,\epsilon)} \Phi \Delta f$$

For $\int_{\mathbb{R}^n \setminus B(0,\epsilon)} \Phi \Delta f$ now integrate by parts to get:

$$\Delta u = \int_{\mathbb{R}^n \setminus B(0,\epsilon)} \Phi \Delta f = \underbrace{\int_{\partial B(0,\epsilon)} \text{Stuff}}_{\text{Show small}} - \int_{\mathbb{R}^n \setminus B(0,\epsilon)} D\Phi \cdot Df$$

And finally, integrate by parts again in the second term to get:

$$\Delta u = -\int_{\mathbb{R}^n \setminus B(0,\epsilon)} D\Phi \cdot Df = - \underbrace{\int_{\partial B(0,\epsilon)} f}_{\text{Exactly what we want}} + \int_{\mathbb{R}^n \setminus B(0,\epsilon)} \underbrace{\Delta \Phi}_{0} f = -\int_{\partial B(0,\epsilon)} f$$

And you just show that the right-hand-side goes to $-f$, so you get $\Delta u = -f$, so $-\Delta u = f$ $\square$