LECTURE 3: MEAN VALUE FORMULA AND CONSEQUENCES

Readings:

- Section 1 of the lecture notes on Change of Variables
- Section 2.2.2: Mean Value Formulas (page 25-26)
- Section 2.2.3a: Strong Maximum Principle, Uniqueness (page 27-28)
- Section 2.2.3f: Harnack's Inequality (pages 32-33)

This week is all about the mean value formula and its incredible consequences!

1. Review: Change of Variables

Video: What is a Jacobian? (This video doesn't cover exactly what's below, but it has the same motivation etc.)

Let me remind you how to do a change of variables from Math 2E. First, let's review u-sub from Math 2B so that you can really compare how similar the two techniques are.

Date: Monday, April 13, 2020.

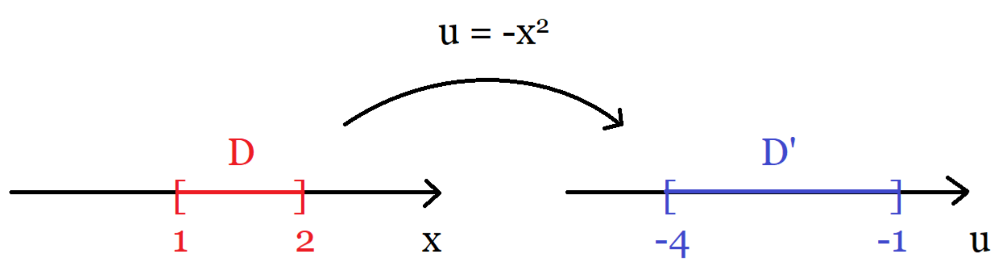
Example:

Evaluate $\int_1^2 e^{-x^2}(-2x)dx$

(1) Let $u = -x^2$

(2) **Endpoints:** $u(1) = -1, u(2) = -4$.

So u turns $D = [1,2]$ into $D' = \cancel{[-1,-4]} = [-4,-1]$.



(3) **du**: Beware of the absolute value! (makes sense, du should be positive)

$$du = \left|\frac{du}{dx}\right| dx = |-2x| dx = 2x dx \Rightarrow -2x dx = -du$$

(4) **Integrate**

$$\begin{aligned}\int_1^2 e^{-x^2}(-2x)dx &= \int_{[1,2]} e^{-x^2}(-2x)dx \\ &= \int_D e^{-x^2}(-2x)dx \\ &= \int_{D'} e^u(-du) \\ &= -\int_{[-4,-1]} e^u du \\ &= -\int_{-4}^{-1} e^u du \\ &= e^{-4} - e^{-1}\end{aligned}$$

Now let's do the Math 2E version:

Example:

Show

$$\int_{B(x,r)} u(y)dy = r^n \int_{B(0,1)} u(x+rz)dz$$

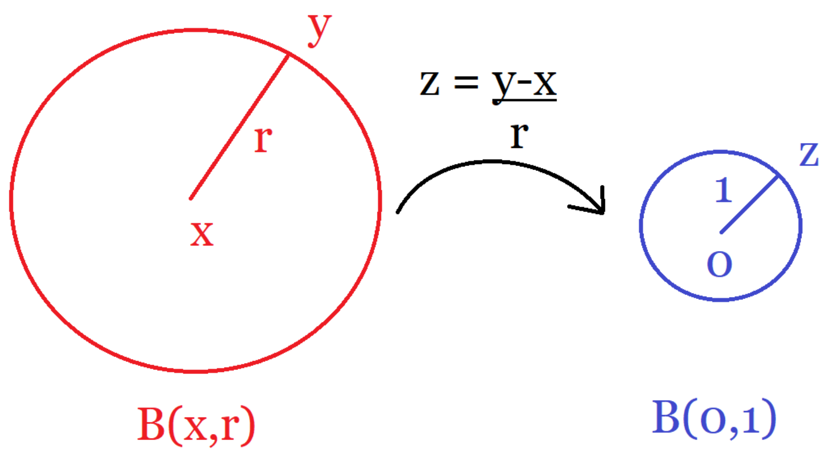
(This is a key ingredient in the proof of the mean value formula below)

(1) Let

$$z = \frac{y-x}{r} = \left(\frac{y_1-x_1}{r}, \frac{y_2-x_2}{r}, \cdots, \frac{y_n-x_n}{r}\right) = (z_1, \cdots, z_n)$$

Then $y = x + rz$

(2) $z(B(x,r)) = B(0,1)$, that is, z maps $B(x,r)$ to $B(0,1)$ (makes sense, the $-x$ in $y-x$ shifts the center from x to 0 and the $\frac{1}{r}$ makes the radius 1)



$$dz = \underbrace{\frac{dz}{dy}}_{?} dy$$

A natural analog of $\frac{dz}{dy}$ would be

$$\frac{dz}{dy} = \begin{bmatrix} \frac{\partial z_1}{\partial y_1} & \cdots & \frac{\partial z_1}{\partial y_n} \\ \vdots & & \vdots \\ \frac{\partial z_n}{\partial y_1} & \cdots & \frac{\partial z_n}{\partial y_n} \end{bmatrix} = \begin{bmatrix} \frac{1}{r} & \cdots & 0 \\ 0 & \vdots & 0 \\ 0 & \cdots & \frac{1}{r} \end{bmatrix}$$

Except we need a scalar instead of a matrix.

Correct Answer:

$$dz = \left| det \begin{bmatrix} \frac{1}{r} & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \frac{1}{r} \end{bmatrix} \right| dy = \frac{1}{r^n} dy$$

Therefore $dy = r^n dz$

(4) Finally, we then get:

$$\int_{B(x,r)} u(y)dy = \int_{B(0,1)} u(x + rz)r^n dz$$

2. The Mean Value Formula

Reading: Section 2.2.2: Mean Value Formulas (page 25-26)

Video: Laplace Mean Value Formula

The **most** important property of Laplace's equation!

Mean Value Formula:

If $\Delta u = 0$, then for any x and $r > 0$ we have

$$⨍_{B(x,r)} u(y)dy = u(x)$$

$$⨍_{\partial B(x,r)} u(y)dS(y) = u(x)$$

In other words, the average value of u over any ball (or sphere) is the value at the center of the ball! In other words, it is *easy* to find the average value of u here.

Note: This only works for the ball, **NOT** for other surfaces!

Proof of (2): Fix x and define

$$\phi(r) = \fint_{\partial B(x,r)} u(y)dS(y) = \frac{\int_{\partial B(x,r)} u(y)dS(y)}{|\partial B(x,r)|}$$

Problem: We cannot directly differentiate this because the domain of integration $\partial B(x,r)$ depends on r.

Solution: Use the change of variables $z = \frac{y-x}{r}$ and using the technique of the previous problem, we get:

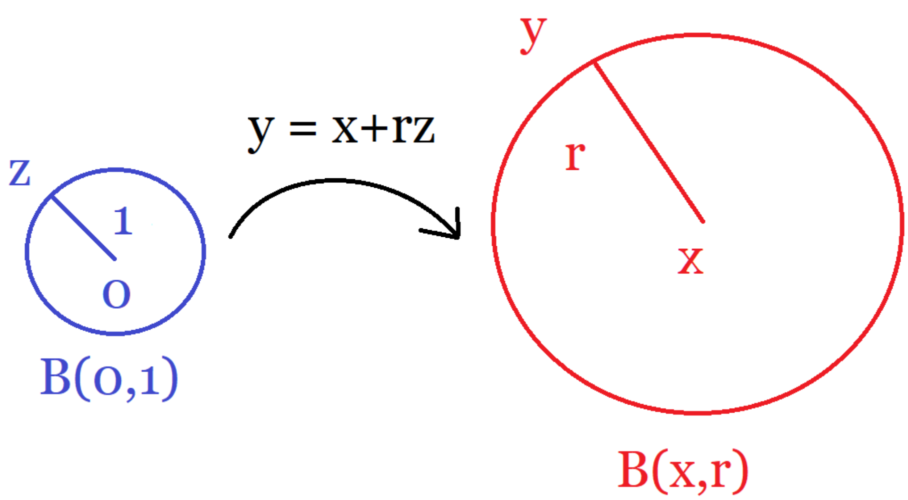
$$\begin{aligned}\phi(r) =& \frac{r^{n-1}\int_{\partial B(0,1)} u(x+rz)dS(z)}{|\partial B(x,r)|}\\ =& \frac{\cancel{r^{n-1}}\int_{\partial B(0,1)} u(x+rz)dS(z)}{n\alpha(n)\cancel{r^{n-1}}}\\ =& \frac{1}{n\alpha(n)}\int_{\partial B(0,1)} u(x+rz)dS(z)\end{aligned}$$

Note: We get r^{n-1} instead of r^n because $\partial B(x,r)$ is $n-1$ dimensional (before we had $B(x,r)$ which was n dimensional)

Since the domain doesn't depend on r, we can differentiate ϕ:

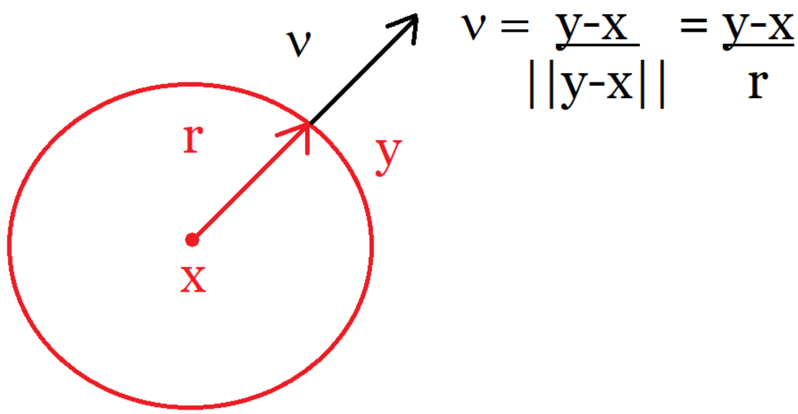
$$\phi'(r) = \frac{\int_{\partial B(0,1)} Du(x+rz)\cdot z dS(z)}{n\alpha(n)}$$

Now change variables back: $y = x + rz$, which transforms $B(0,1)$ back into $B(x,r)$:



We then get

$$
\begin{aligned}
\phi'(r) =& \frac{1}{n\alpha(n)} \int_{\partial B(x,r)} Du(y) \cdot \underbrace{\left(\frac{y-x}{r}\right)}_{\nu} \frac{1}{r^{n-1}} dS(y) \\
=& \frac{1}{n\alpha(n) r^{n-1}} \int_{\partial B(x,r)} Du(y) \cdot \nu dS(y) \\
=& \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} \frac{\partial u}{\partial \nu}
\end{aligned}
$$



Recall: Integration by parts

$$\int_U (\Delta u) v dx = \int_{\partial U} \frac{\partial u}{\partial \nu} v - \int_U Du \cdot Dv$$

With $v = 1$ this simply becomes

$$\int_U \Delta u = \int_{\partial U} \frac{\partial u}{\partial \nu}$$

Therefore:

$$\phi'(r) = \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} \frac{\partial u}{\partial \nu} = \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} \underbrace{\Delta u}_{0} = 0$$

Hence $\phi(r) = \fint_{\partial B(x,r)} u(y) dS(y)$ is constant, and letting $r \to 0$, we get

$$\fint_{\partial B(x,r)} u(y) dS(y) = \phi(r) = \lim_{r \to 0} \phi(r) = u(x)$$

(The last part uses continuity of u and I think is an exercise in the suggested HW)

Proof of (1): Much easier! Just use (2) and the polar coordinates formula!

$$
\begin{aligned}
\frac{\int_{B(x,r)} u(y)}{|B(x,r)|} &= \frac{1}{\alpha(n)r^n} \int_0^r \int_{\partial B(x,t)} u(y) dS(y) dt \\
&= \frac{1}{\alpha(n)r^n} \int_0^r \left(\frac{\int_{\partial B(x,t)} u(y)}{|\partial B(x,t)|} |\partial B(x,t)| \right) \\
&= \frac{1}{\alpha(n)r^n} \int_0^r u(x) n\alpha(n) t^{n-1} dt \qquad \text{Using (2)} \\
&= \frac{1}{\alpha(n)r^n} u(x) n\alpha(n) \frac{r^n}{n} \\
&= u(x) \quad \square
\end{aligned}
$$

Note: In fact the mean value formula is equivalent to $\Delta u = 0$ (see book)

3. Maximum Principle

Reading: Section 2.2.3a: Strong Maximum Principle, Uniqueness (page 27-28)

The rest of today is just about applications of the mean value formula, starting with the Maximum Principle

Maximum Principle:

If $\Delta u = 0$, then

(1) Weak:
$$\max_{\overline{U}} u = \max_{\partial U} u$$
(but could be attained inside U)

(2) Strong: $\max_{\overline{U}} u$ is attained **only** on ∂U (unless u is constant)

From this, we can deduce uniqueness of solutions of Poisson's equation $-\Delta u = f$

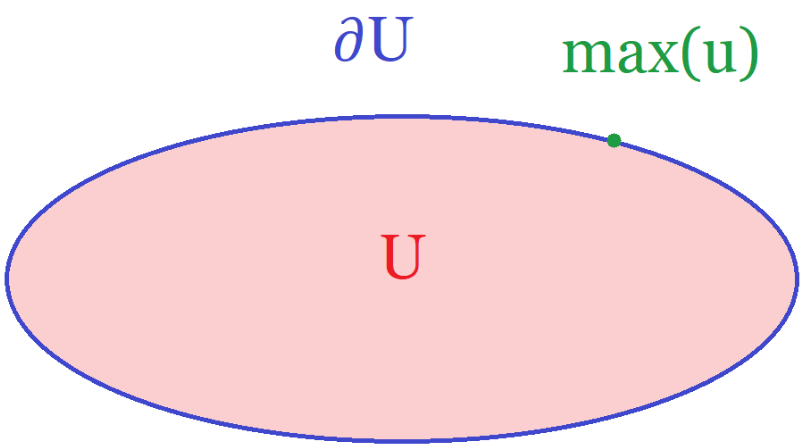
See Proofs in the book

4. Positivity

Positivity:

Suppose u satisfies
$$\begin{cases} \Delta u = 0 & \text{in } U \\ u = g & \text{on } \partial U \end{cases}$$
Where $g \geq 0$ and $g \not\equiv 0$, that is g is positive *somewhere*

Then $u > 0$ *everywhere* in U



Proof:

By the weak maximum principle (with min instead of max)

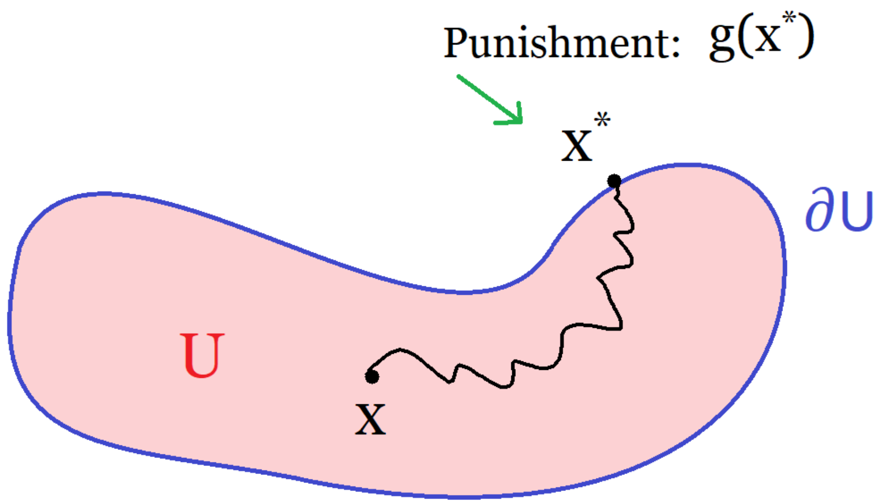
$$\min_{\overline{U}} u = \min_{\partial U} u = \min_{\partial U} g \geq 0$$

Hence $u \geq 0$.

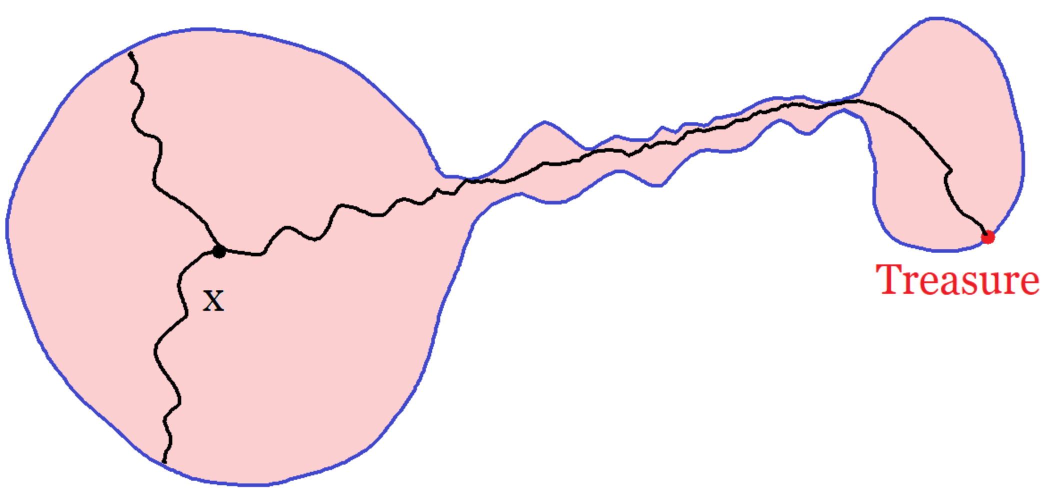
But if $u(x^{\star}) = 0$ for some $x^{\star} \in U$, then u has a minimum inside U, which implies $u \equiv 0$ in U (and hence in $\overline{U}$ by continuity) and this implies $g \equiv 0 \Rightarrow\Leftarrow$ $\square$

Awesome Application: Remember the interpretation of Laplace's equation in terms of Brownian Motion (from Week 1). Namely

$u(x)$ = Expected gain/loss starting at x



Now suppose U is a very weird domain and g is zero everywhere, except for a tiny point where it's positive (imagine there is a treasure there):



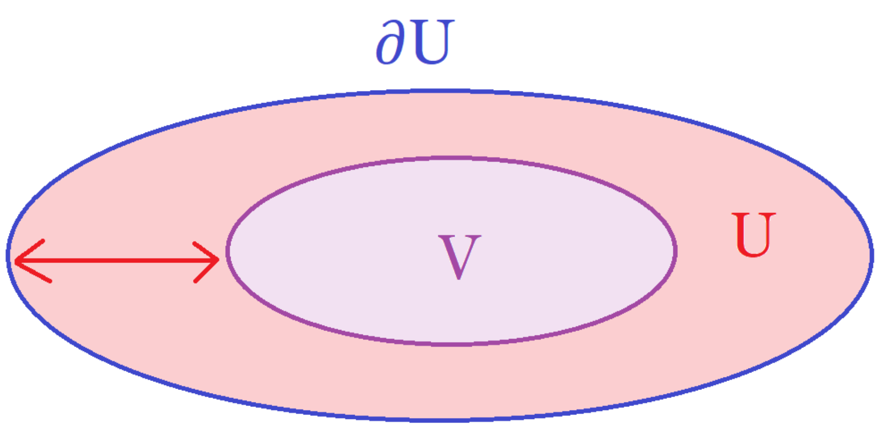
Then positivity says that $u > 0$ everywhere, which implies that, *no matter where you start*, it not only possible to reach that treasure, but there's a positive probability of doing so! (Because if the probability of reaching the point were 0, then the average value would be 0 as well since $g = 0$ everywhere else)

5. Harnack's Inequality

Reading: Section 2.2.3f: Harnack's Inequality (pages 32-33)

Very strange statement, but it's kind of a regularizing effect of Laplace's Equation.

Note: $V \subset\subset U$ just means that there is some space (or wiggle room) between V and ∂U.



Harnack's Inequality:

The is a constant C depending *only* on V (and not on u) such that for all u, if $\Delta u = 0$ and $u \geq 0$, then:

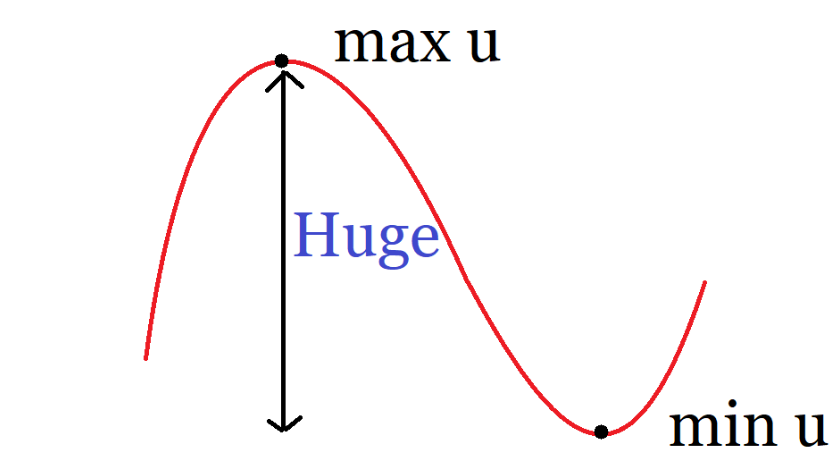
$$\max_V u \leq C \min_V u$$

Really think of C as just being a constant. For example, if V is a ball, think $C = 5$.

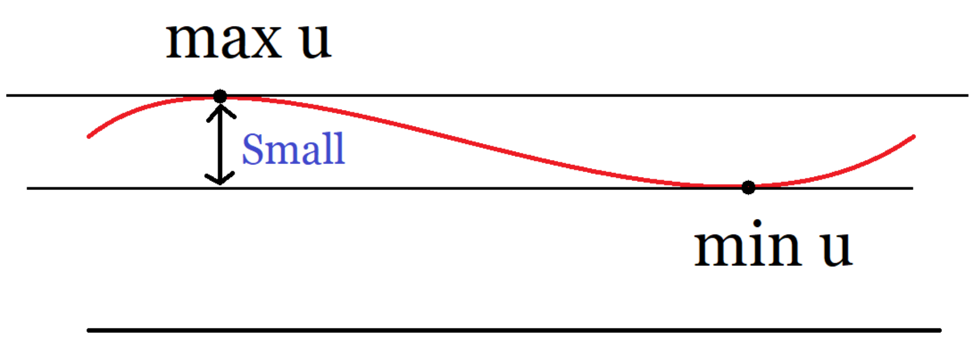
What this is saying is that if the minimum of u is small, then the maximum of u is small too.

For example, say $C = 5$ and the smallest value of u is 2, this is saying that the largest value of u cannot be 100 because otherwise you'd get $100 \leq 5(2) = 10$.

So harmonic functions generally don't look like this:



But rather like this:



(Again, see proof in the book)