LECTURE 4: MORE CONSEQUENCES AND ENERGY METHODS

Readings:

- (Optional) Section 2.2.3c: Local Estimates For Harmonic Functions (page 29, only the case $k = 1$)
- Section 2.2.3d: Liouville's Theorem (page 30)
- Appendix C.5: Convolution and Smoothing (pages 713-714, only the definitions)
- Section 2.2.3b: Regularity (page 28)
- Section 2.2.5: Energy Methods (pages 41-43)
- Calculus of Variations (Section 6 in those notes)

Reminder: This week is all about more consequences of Laplace's equation, as well as an introduction to energy methods.

1. Liouville's Theorem

Readings: Section 2.2.3d: Liouville's Theorem (page 30) and also (Optional) Section 2.2.3c: Local Estimates For Harmonic Functions (page 29, only the case $k = 1$)

Date: Monday, April 13, 2020.

The following result seems almost too good to be true (but it is good **AND** it is true):

Liouville's Theorem:

If $\Delta u = 0$ in $\mathbb{R}^n$ and u is bounded, then u must be constant!

In other words, any (non-constant) harmonic function must blow up somewhere (possibly at ∞). For for instance, the fact that $\Phi(x) = \frac{C}{|x|^{n-2}}$ blows up at 0 isn't an anomaly, but a feature that most harmonic functions share!

Example: In 2 dimensions, $u(x,y) = x^2 - y^2$ solves Laplace's equation, and notice that u is unbounded! Similarly with $u(x,y) = e^x \sin(y)$

Note: If this theorem sounds familiar to you, then you're correct! In fact, in Complex Analysis, Liouville's Theorem says that if $f(z)$ is a bounded holomorphic function, then f is constant. And this is not a surprise because remember that if f is holomorphic, then $Re(f)$ and $Im(f)$ solve Laplace's equation!

To prove this, we need an estimate in section 2.2.3c, which you can prove in return prove using the mean-value formula. I cannot emphasize how important estimates are in PDE. They are the main tools that allow us to prove cool PDE facts (like here to prove Liouville's Theorem)

Definition:

$$\|u\|_{L^1(U)} = \int_U |u(x)|\, dx$$

(Tells you how 'big' u is, but in terms of integrals)

Decay Estimate:

If $\Delta u = 0$, then

$$|Du(x)| \leq \frac{\sqrt{n}C}{r^{n+1}} \|u\|_{L^1 B(x,r)}$$

(See proof in the book)

Intuitively: This estimate is saying that in $n = 1$ dimensions, $Du = u'$ decays at most like $\frac{1}{r^2}$, in 2 dimensions, Du decays at most like $\frac{1}{r^3}$, and in 3 dimensions, Du decays at most like $\frac{1}{r^4}$ etc. So Du must be small if r is large, and no bigger than $\frac{C}{r^{n+1}}$.

Proof of Liouville: Suppose u is bounded, that is there is some $C > 0$ such that $|u(y)| \leq C$ for all y.

Then, first of all:

$$\begin{aligned} \|u\|_{L^1(B(x,r))} &= \int_{B(x,r)} \underbrace{|u(y)|}_{\leq C} dy \\ &\leq C \int_{B(x,r)} 1 dy \\ &= C\,|B(x,r)| \\ &= C\alpha(n) r^n \end{aligned}$$

Therefore, by our estimate:

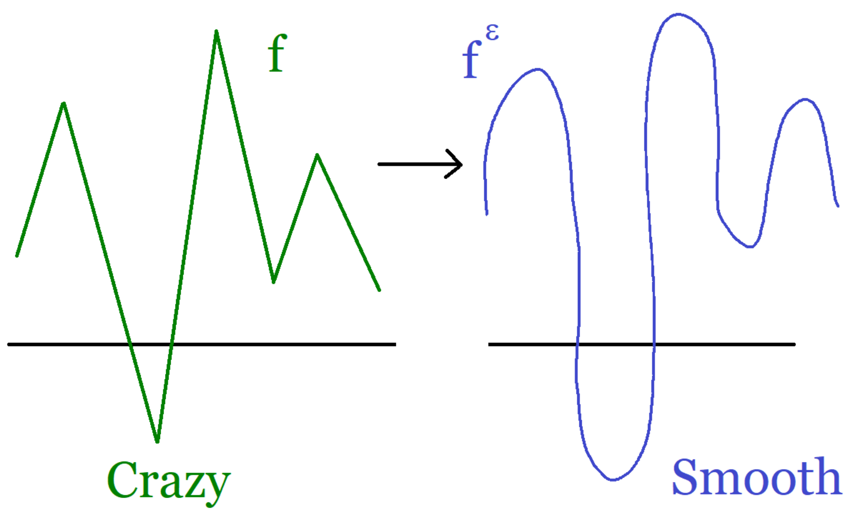
$$
\begin{aligned}
|Du(x)| &\leq \frac{C\sqrt{n}}{r^{n+1}} \|u\|_{L^1 B(x,r)} \\
&\leq \frac{C\sqrt{n}}{r^{n+1}} C\alpha(n) r^n \\
&= \frac{C\sqrt{n}\alpha(n)}{r} \qquad \text{(Different C)} \\
&\to 0 \qquad (\text{As } r \to \infty)
\end{aligned}
$$

Hence $Du(x) = 0$, and so u is constant $\square$

2. Mollifiers

Reading: Appendix C.5: Convolution and Smoothing (pages 713-714, only the definitions)

Main Idea: Given a **WILD** function f, is it possible to approximate it by a **SMOOTH** function f^ϵ that is "close" to f? (whatever that means)



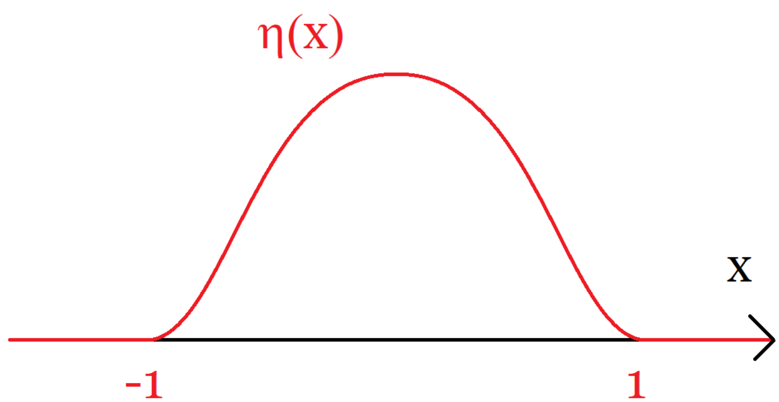
The amazing fact is that we can do this, using convolution (= analysis multiplication). First we need a super smooth function:

STEP 1: Bump Function

Definition:

$$\eta(x) = \begin{cases} Ce^{\frac{1}{|x|^2-1}} & \text{If } |x| < 1 \\ 0 & \text{If } |x| \geq 1 \end{cases}$$

This is called a *bump* function. C is a constant such that $\int \eta(x)dx = 1$.

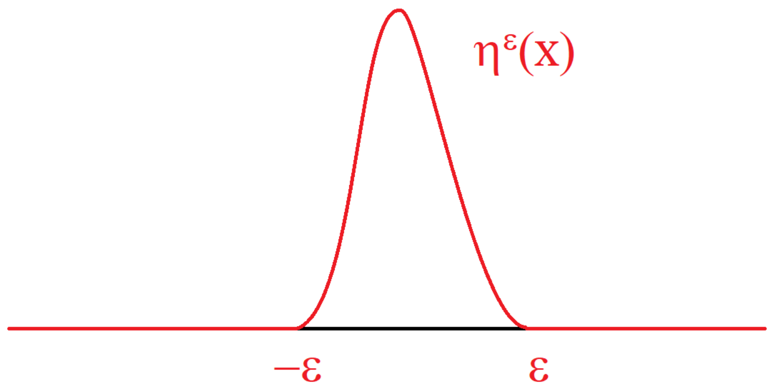


STEP 2: Spike it up! (= Make the bump function high and thin)

Definition:

$$\eta^{\epsilon}(x) = \frac{1}{\epsilon^n}\eta\left(\frac{x}{\epsilon}\right)$$

For instance, if $\eta = 0.01$, then $\eta^{\epsilon}(x) = 100\eta(100x)$



(The $\frac{1}{\epsilon}^n$ factor is there so that the integral of η^ϵ is the same as the integral of η). Notice in fact that as $\epsilon \to 0$, then $\eta^\epsilon \to \delta_0$, so the next result shouldn't be surprising!

STEP 3: Mollifier

Given f (WILD), define:

Mollifier:

$$f^\epsilon(x) = \eta^\epsilon \star f = \int \eta^\epsilon(y) f(x-y) dy$$

Facts:

(1) $f^\epsilon \in C^\infty$ for all ϵ

(2) $f^\epsilon \to f$ pointwise as $\epsilon \to 0$

Why? The precise proofs are in the book, but intuitively (1) follows because:

$$(f^\epsilon)' = \left(\int \eta^\epsilon(y) f(x-y) \right)' = \int (\eta^\epsilon(y))' f(x-y)$$

Notice that all the derivatives fall on η^ϵ, so since η^ϵ is infinitely differentiable, so is f^ϵ
And (2) follows because $\eta^\epsilon \to \delta_0$ and therefore

$$f^\epsilon = f \star \eta^\epsilon \to f \star \delta_0 = f(x)$$

But of course all of this is non-rigorous, and you should really check out the book (if you want) to see hwo to make those proofs rigorous

3. Smoothness

Reading: Section 2.2.3b: Regularity (page 28)

Using mollifiers and the mean value formula, you can then show a really beautiful and unexpected result:

Smoothness:

If $u \in C^2(U)$ solves $\Delta u = 0$, then $u \in C^\infty(U)$

In other words, any solution of $\Delta u = 0$ is **INFINITELY** differentiable, wow!!!

The main idea is to show, using the mean-value formula, that u must be equal to its mollifier u^ϵ for all ϵ. Since u^ϵ is C^∞ (by the previous section), it follows that u must be C^∞. The details of the proof are in the book.

4. Energy Methods

Reading: Section 2.2.5: Energy Methods (pages 41-43)

There are two important classes of methods in PDEs:

(1) **Maximum Principle Methods**, which we've seen last time with uniqueness and positivity

(2) **Energy Methods**, which are based on integration by parts (see below)

Here's an example of a result that you can prove using energy methods

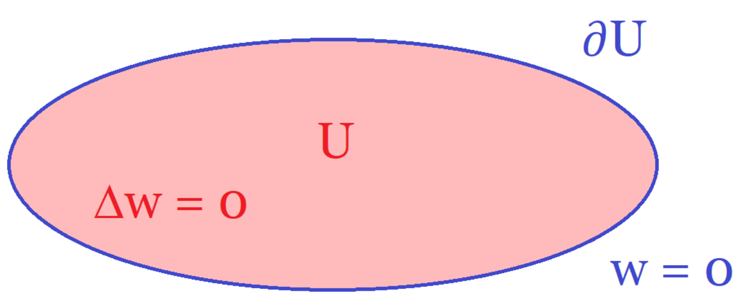
Uniqueness:

There exists at most one solution of

$$\begin{cases} -\Delta u = f & \text{(in } U) \\ u = g & \text{(on } \partial U) \end{cases}$$

Proof: Let u and v be two solutions, and let $w = u - v$, then w solves:

$$\begin{cases} -\Delta w = 0 \\ w = 0 \end{cases}$$



Energy Method: Multiply $-\Delta w$ by a clever function (here w) and integrate by parts:

$$\int_U (\underbrace{-\Delta w}_{0}) w \overset{IBP}{=} \int_{\partial U} - \left(\frac{\partial w}{\partial \nu}\right) \underbrace{w}_{0} + \int_U Dw \cdot Dw$$
$$= \int_U |Dw|^2$$

Therefore $\int_U |Dw|^2 = 0$ but this implies $Dw \equiv 0$ so $w \equiv C$. But since $w = 0$ on ∂U, this implies that $C = 0$, so $w \equiv 0$ everywhere.

Hence $w = u - v \equiv 0$, so $u \equiv v$ $\square$

Note: The choice of multiplying by w seems a bit random, and it is! In fact, a lot of modern PDE research is devoted to simply figuring out which function to multiply your PDE with (Here w) to get an interesting result!

5. Dirichlet's Principle

Video: Calculus of Variations (doesn't cover *exactly* the same material, but it's a good introduction)

As a further example of energy methods, let's discuss an elegant result which connects Poisson's equation with minimizers of an energy (which explains why this is called an *energy* method).

Definition:

$$I[w] = \int_U \frac{1}{2} |Dw|^2 - wf$$

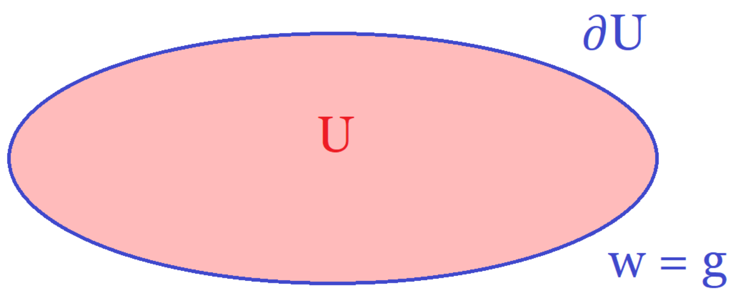
This is called the *energy functional.* Notice that, in terms of physics, this is the sum of the kinetic energy $\frac{1}{2}(\text{ Speed }^2)$ and the potential energy $-wf$ (Minus because the force is pointing downwards)

That said, w is not a random function, but needs to be part of an admissible class:

Definition:

$$\mathcal{A} = \{w \,|\, w = g \text{ on } \partial U \}$$

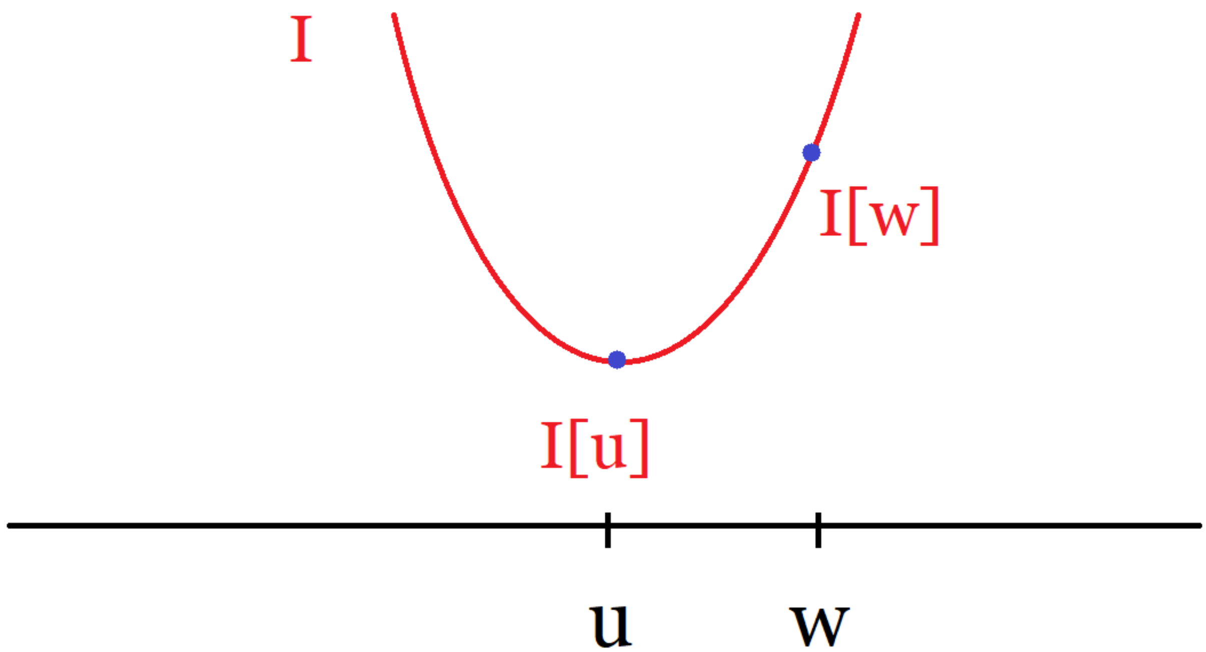
That is, *only* focus on functions that are equal to g on ∂U (where g is a **fixed** function)



Dirichlet's Principle:

$$u \text{ solves } -\Delta u = f \Leftrightarrow u \text{ minimizes } I[u]$$

So among *all* the functions $w \in \mathcal{A}$, it's really u that makes $I[w]$ the smallest. So the energy profile of I really looks as follows:



For the proof, again see the book. It's a very beautiful proof, in my opinion, one of the most beautiful proofs in the book!

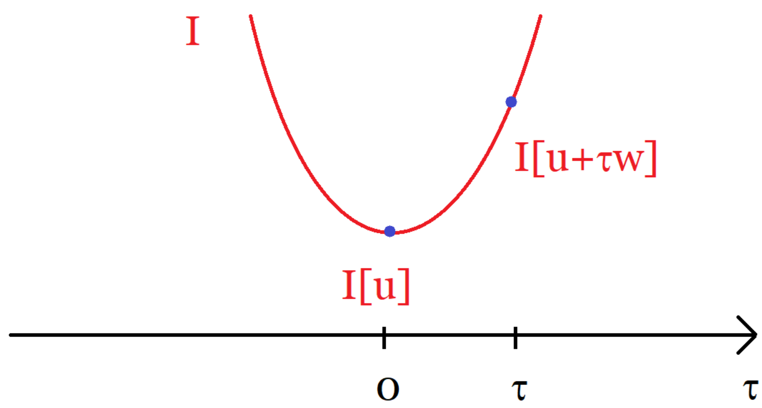
Main Idea:

($\Rightarrow$) Use Integration by Parts

($\Leftarrow$) Consider $i(\tau) = I[u + \tau v]$.

Notice that i is a function of *one* real variable τ.

Moreover, i has a minimum precisely at $\tau = 0$ because I has a minimum at $u = u + 0v$ but doesn't have a minimum at $u + \tau v$, as in the following picture



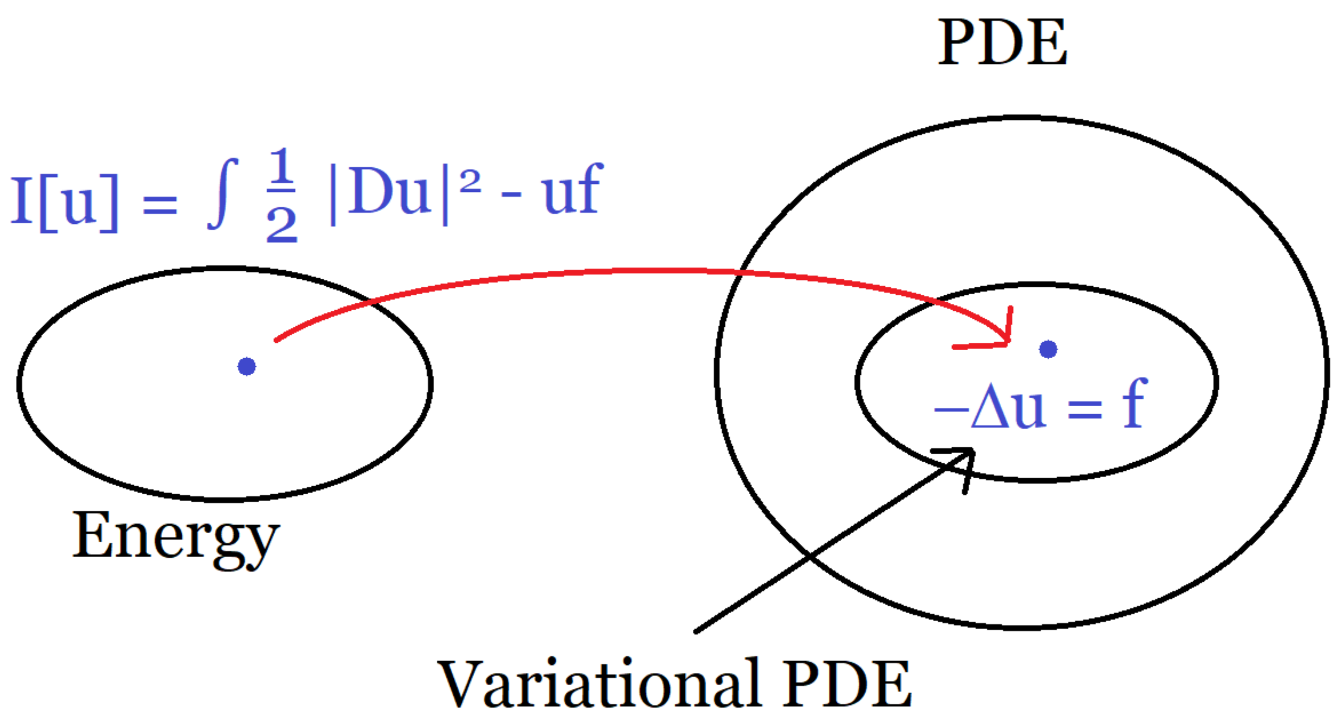
Therefore from Math 2A, you get $i'(0) = 0$, and from this you can deduce that u solves $-\Delta u = f$ (see book)

What makes this so elegant is that you transform a complicated energy problem into an easy calculus problem!

6. Calculus of Variations

Dirichlet's Principle is a special case of calculus of variations (one of my specializations actually). In layman's terms, Calculus of Variations relates solutions of PDEs with minimizers of energies.

Here, for instance, we related the Dirichlet energy $I[u] = \int_U \frac{1}{2} |Du|^2 - uf$ with the Poisson equation $-\Delta u = f$.



In fact, this is always true, and is the cornerstone of the field of calculus of variations:

Fact:

If u minimizes the energy $I[u]$, then u solves a PDE, called the Euler-Lagrange equation

Example:

If $I[u] = \int \frac{1}{2}|Du|^2 - uf$, then the Euler-Lagrange equation is $-\Delta u = f$

In practice, it's **HARD** to solve a PDE, but **EASY** to minimize an energy, so **IF** you can write a PDE as a minimizer of an energy, then it's **GOOD** and the PDE is called **VARIATIONAL**.

And variational PDEs are good because it means you actually have a shot at solving it; there's a well-established theory of minimizers.

Example:

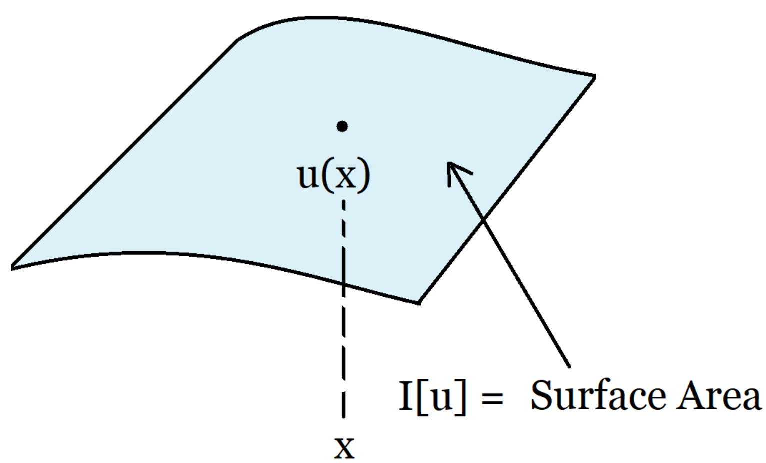
The minimal surface PDE is

$$\sum_{i=1}^{n}\left(\frac{u_{x_i}}{\sqrt{1+|Du|^2}}\right)_{x_i}=0$$

As horrible as it sounds, it's actually variational! u is actually the minimizer of

$$I[u]=\int_U \sqrt{1+|Du|^2}dx$$

Which is much easier because $\int \sqrt{1+|Du|^2}$ is just the surface area of the graph of u !



So instead of solving the PDE, all you have to figure out is: Which graphs (with given boundary conditions) have the smallest surface area? Like a soap film for instance.

Note: The following Wiki article has **beautiful** pictures of minimal surfaces. Check it out if you want to be amazed: Minimal Surfaces. (Click on each example for a picture)

Note: We'll see energy methods again with the heat equation and the wave equation, so you can really see how powerful this class of methods is.

Congratulations, we're officially done with Laplace's equation! In the next 3 lectures, we'll discuss the heat equation.