

LECTURE 5: THE HEAT EQUATION

Readings:

- Physical Interpretation of the heat equation (page 44)
- Applications of the Heat Equation (section 2 below)
- Section 2.3.1a: Derivation of the Fundamental Solution (pages 45-46)
- Gaussian Integral (section 4 below)
- Section 2.3.1b: Initial-Value Problem (pages 47-49)

In the next 3 weeks, we'll talk about the heat equation, which is a close cousin of Laplace's equation. In fact, both of them share very similar properties

Heat Equation:

$$u_t = \Delta u$$

1. DERIVATION OF THE HEAT EQUATION

Reading: Physical Interpretation of the heat equation (page 44)

The derivation of the heat equation is very similar to the derivation of Laplace's equation (The derivation of Laplace's equation can be found

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in this video).

For Laplace's equation, we considered a fluid F in equilibrium, meaning that $\int_{\partial V} F \cdot d\nu = 0$ and used the divergence theorem.

Here the difference is that the fluid is not in equilibrium any more, but instead evolves according to the net flux, meaning that:

$$\underbrace{\frac{d}{dt} \int_V u dx}_{\text{Time Evolution}} = - \underbrace{\int_{\partial V} F \cdot d\nu}_{\text{Net Flux}}$$

But just as before, using the divergence theorem trick and assuming $F = -aDu$, we ultimately obtain

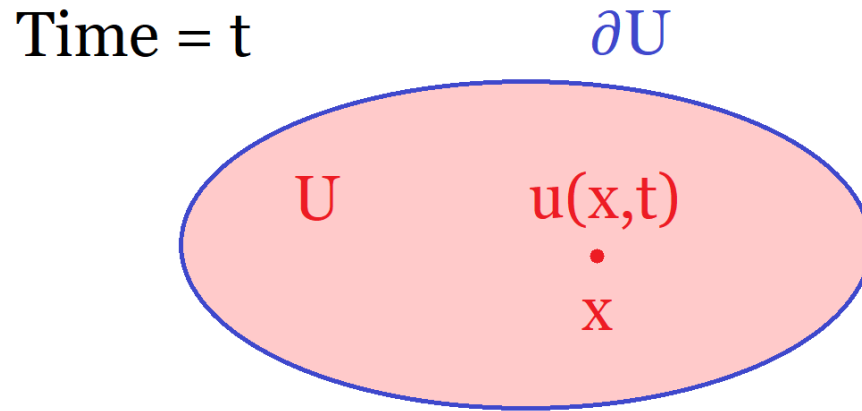
$$u_t = a\Delta u$$

Which is the heat equation for $a = 1$.

2. APPLICATIONS OF THE HEAT EQUATION

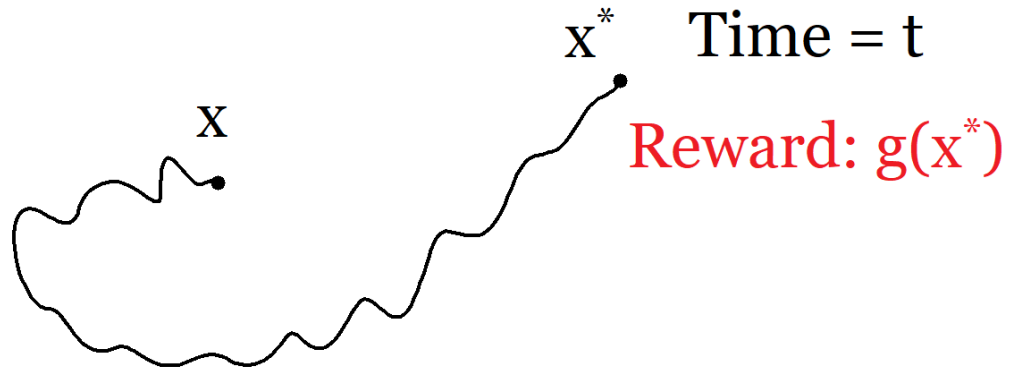
Just like with Laplace, there are many beautiful applications of the heat equation:

- (1) **Particle Diffusion:** $u(x, t)$ is the temperature of a metal plate at position x and time t .

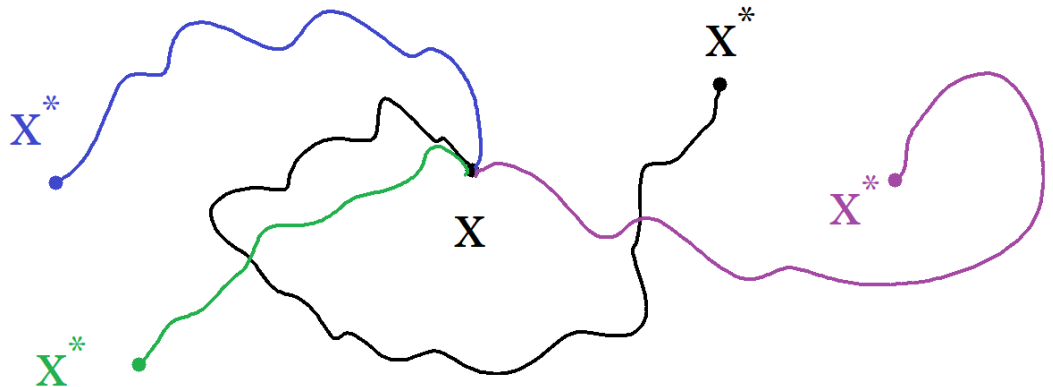
**Remarks:**

- (i) Compare this with the solution $u(x)$ of Laplace's equation, which gave the temperature of a metal plate at x , but after a loooong time.
 - (ii) If u is a chemical density of a particle, then $u(x, t)$ = chemical concentration of the particle at x and time t .
 - (iii) Sometimes, the heat equation is also called the diffusion equation, measures how a particle diffuses (think for instance as putting a blue dye in a glass of water)
- (2) **Chemical Reactions:** Can model a simple chemical reaction $A \rightleftharpoons B$ with a system of heat equations. In fact, that's what my PhD research was all about! Feel free to check out the following video in case you're interested: The PDE that gave me the PhD.
- (3) **Brownian Motion:** Remember the example with Laplace about Brownian motion and hitting a wall? There's a fancier version with the heat equation.

Suppose you start at x , do your usual Brownian motion, and at a *fixed* time t , I tell you to stop. Then you're at the point x^* , and I give you a reward $g(x^*)$



Again, this is a random event, so it makes sense to take the average/expected value.



Let $u(x, t)$ = Average (= Expected) gain/loss at time t starting at x

Then $u_t = \Delta u$

- (4) **Schrödinger's Equation:** $u_t = i\Delta u$, which is the foundation of quantum mechanics
- (5) **Finance:** The Black-Scholes equation pricing model, which is used to predict the stock market is a Stochastic (= Random) Differential Equation that can be transformed into the heat equation
- (6) Used in **Machine Learning**
- (7) **Geometry:** The Ricci Flow is a heat equation on surfaces, which Perelman used to solve the Poincaré Conjecture
- (8) **Fourier Series:** The Heat Equation was the equation Fourier studied when he discovered his series!

3. FUNDAMENTAL SOLUTION

Readings: Section 2.3.1a: Derivation of the Fundamental Solution (pages 45-47)

Video: Fundamental Solution of the Heat Equation

Motivation: In $\Delta u = 0$, the quantity $|x|^2 = (x_1)^2 + \cdots + (x_n)^2$ played an important role. And in fact radial solutions satisfy $|x|^2 = \text{Constant}$.

By analogy, in $u_t = \Delta u$, the quantity $t = |x|^2$ might seem to play a role. But:

$$t = |x|^2 \Rightarrow \frac{|x|^2}{t} = 1 \Rightarrow \left(\frac{|x|}{\sqrt{t}} \right)^2 = 1$$

First Guess: Guess $u(x, t) = v\left(\frac{x}{\sqrt{t}}\right)$. This doesn't quite work, because it turns out that u blows up near $t = 0$.

STEP 1: Better Guess:

$$u(x, t) = \frac{1}{t^{\frac{n}{2}}} v\left(\frac{x}{\sqrt{t}}\right)$$

For some $v = v(y) : \mathbb{R}^n \rightarrow \mathbb{R}$ TBA The following result seems almost too good to be true (but it is good **AND** it is true):

Plug the above formulation into the PDE $u_t = \Delta u$ to get

$$\left(\frac{n}{2}\right) v + \frac{1}{2} Dv \cdot y + \Delta v = 0$$

This is **GOOD** in the sense that there's no more t .

But this is also **BAD** in the sense that it's still a PDE!

STEP 2: Just like for Laplace, guess v is radial, that is $v(y) = w(|y|) = w(r)$ for some $w : \mathbb{R}^+ \rightarrow \mathbb{R}$.

Then plugging this formulation in the PDE above, we get:

$$\frac{1}{2} (w'r + wn) + w'' + \left(\frac{n-1}{r} \right) w' = 0$$

And the beautiful thing is that we can explicitly solve it!

STEP 3: Solve the above ODE, which gives

$$w(r) = Ce^{-\frac{r^2}{4}}$$

Therefore:

$$v(y) = w(|y|) = Ce^{-\frac{|y|^2}{4}}$$

Hence

$$u(x, t) = \frac{1}{t^{\frac{n}{2}}} v\left(\frac{x}{\sqrt{t}}\right) = \frac{C}{t^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}$$

Which ultimately gives the fundamental solution:

Fundamental Solution of the Heat Equation:

$$\Phi(x, t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}$$

Note: The constant $\frac{1}{(4\pi t)^{\frac{n}{2}}}$ is chosen so that for all t ,

$$\int_{\mathbb{R}^n} \Phi(x, t) dx = 1$$

So the fundamental solution has constant mass.

4. GAUSSIAN INTEGRAL

Video: Gaussian Integral

Video: Gaussian Integral Playlist

Video: Gauss Cubed

The fact that $\Phi(x, t)$ has integral 1 relies on the following beautiful result due to Gauss

Gaussian Integral:

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

Proof: Let $I = \int_{-\infty}^{\infty} e^{-x^2} dx \geq 0$

But also $I = \int_{-\infty}^{\infty} e^{-y^2} dy$
(doesn't matter which variable we're using; potato potahto)

Multiply:

$$\begin{aligned} I^2 &= (I)(I) \\ &= \left(\int_{-\infty}^{\infty} e^{-x^2} dx \right) \left(\int_{-\infty}^{\infty} e^{-y^2} dy \right) \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{-x^2} dx \right) e^{-y^2} dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2} e^{-y^2} dx dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx dy \end{aligned}$$

$$\begin{aligned}
&= \int_0^{2\pi} \int_0^\infty e^{-r^2} r dr d\theta \\
&= 2\pi \int_0^\infty r e^{-r^2} dr \\
&= 2\pi \left[\left(-\frac{1}{2} \right) e^{-r^2} \right]_{r=0}^{r=\infty} \quad (u\text{-sub} : u = -r^2) \\
&= 2\pi \left(-\frac{1}{2} e^{-\infty} + \frac{1}{2} e^0 \right) \\
&= 2\pi \left(\frac{1}{2} \right) \\
&= \pi
\end{aligned}$$

$$I^2 = \pi \Rightarrow I = \sqrt{\pi} \quad (\text{since } I > 0)$$

Answer:

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

Note: Check out this awesome playlist for 12 ways of evaluating this integral: Gaussian Integral 12 Ways

Note: For a crazy spherical coordinates version, check out: Gauss Cubed

Corollary:

$$\int_{\mathbb{R}^n} \Phi(x, t) dx = 1$$

Proof:

$$\begin{aligned}
\int_{\mathbb{R}^n} \Phi(x, t) dx &= \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4t}} dx \\
&= \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{(x_1)^2 + (x_2)^2 + \dots + (x_n)^2}{4t}} dx \\
&= \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{(x_1)^2}{4t} - \frac{(x_2)^2}{4t} - \dots - \frac{(x_n)^2}{4t}} dx_1 dx_2 \dots dx_n \\
&= \frac{1}{(4\pi t)^{\frac{n}{2}}} \left(\int_{-\infty}^{\infty} e^{-\frac{(x_1)^2}{4t}} dx_1 \right) \dots \left(\int_{-\infty}^{\infty} e^{-\frac{(x_n)^2}{4t}} dx_n \right) \\
&= \frac{1}{(4\pi t)^{\frac{n}{2}}} \left(\sqrt{4t} \int_{-\infty}^{\infty} e^{-(u_1)^2} du_1 \right) \dots \left(\sqrt{4t} \int_{-\infty}^{\infty} e^{-(u_n)^2} du_n \right) \\
&= \frac{1}{(4\pi t)^{\frac{n}{2}}} \left(\sqrt{4t} \right)^n \left(\int_{-\infty}^{\infty} e^{-u^2} du \right)^n \\
&= \frac{1}{(4\pi t)^{\frac{n}{2}}} \left(\sqrt{4t} \right)^n (\sqrt{\pi})^n \\
&= \frac{(\sqrt{4\pi t})^n}{(\sqrt{4\pi t})^n} \\
&= 1
\end{aligned}$$

Note: In the middle, we used the u -sub $u_1 = \frac{x_1}{\sqrt{4\pi}}$, $u_2 = \frac{x_2}{\sqrt{4\pi}}$ etc. And in the step afterwards we used that all the n integrals are the same and equal to $\int_{-\infty}^{\infty} e^{-u^2} du = \sqrt{\pi}$

5. INITIAL VALUE PROBLEM

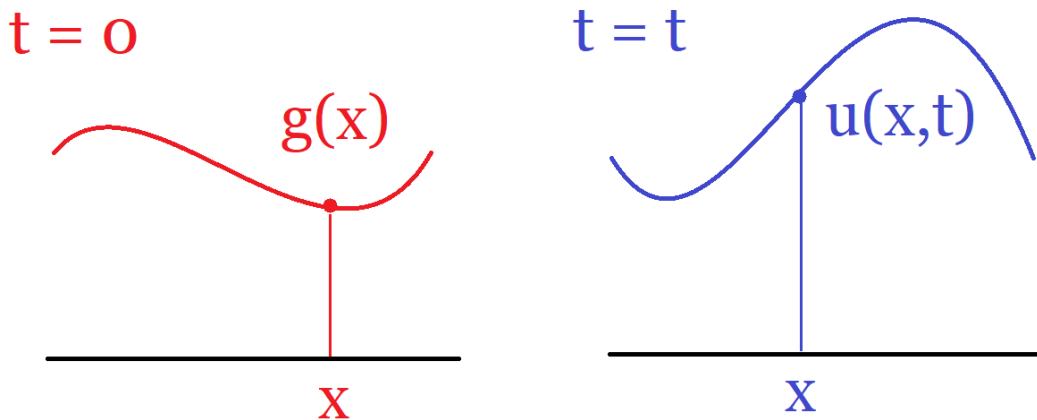
Reading: Section 2.3.1b: Initial Value Problem (pages 47-48)

Video: Initial-Value Problem

Now suppose we would like to solve the heat equation with an initial condition:

$$\begin{cases} u_t - \Delta u = 0 \\ u(x, 0) = g(x) \end{cases}$$

Where $g(x)$ is given.



Motivation: To solve $-\Delta u = f$, we used

$$u = \Phi \star f = \int_{\mathbb{R}^n} \Phi(x - y) f(y) dy$$

The same thing works here:

Initial Value Problem:

$$u(x, t) = \Phi \star g = \int_{\mathbb{R}^n} \Phi(x - y, t) g(y) dy$$

Solves the problem above

Except we have to be more precise by what we mean by $u(x, 0)$ since technically the fundamental solution is not defined at 0

Claim 1: $u \in C^\infty(\mathbb{R}^n \times (0, \infty))$

This is not a problem at all, because for every $t > 0$, we can differentiate the above solution. For example,

$$u_{x_1} = \left(\int_{\mathbb{R}^n} \Phi(x - y, t) g(y) dy \right)_{x_1} = \int_{\mathbb{R}^n} \Phi_{x_1}(x - y, t) g(y) dy$$

(Technically, we used the Dominated Convergence Theorem trick from Lecture 2)

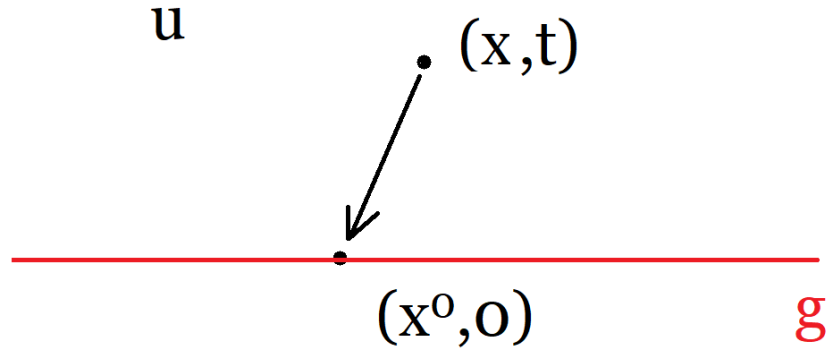
And since Φ is infinitely differentiable (and decays fast to 0), it follows that u is infinitely differentiable

Claim 2: $u_t = \Delta u$ (if $t > 0$)

Also not a problem, since

$$u_t - \Delta u = \int_{\mathbb{R}^n} (\Phi_t - \Delta \Phi) g = \int 0g = 0$$

Claim 3: $\lim_{(x,t) \rightarrow (x^0,0)} u(x,t) = g(x^0)$



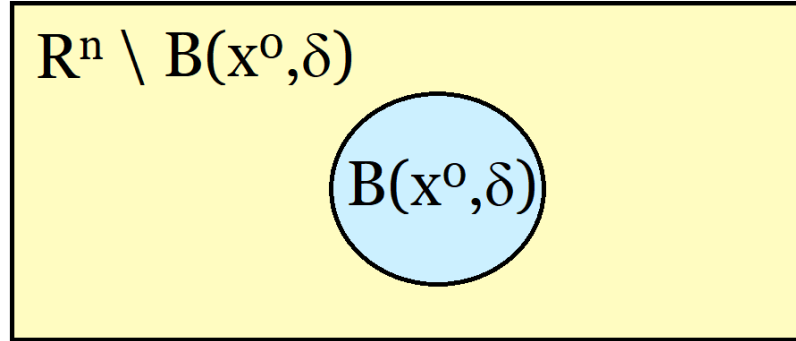
In other words, $u(x^0, 0) = g(x^0)$ (but the above is more rigorous)

This is not obvious, since Φ blows up near 0. The idea is to reason just like we did for Poisson's equation, by doing a little bit of surgery near x^0 .

Main Idea: Show that if $|x - x_0|$ and $|t|$ are small, then $|u(x, t) - g(x^0)|$ is small.

STEP 1: We can write:

$$\begin{aligned} |u(x, t) - g(x^0)| &\leq \int_{B(x^0, \delta)} \Phi(x - y, t) |g(y) - g(x^0)| dx \\ &\quad + \int_{\mathbb{R}^n \setminus B(x^0, \delta)} \Phi(x - y, t) |g(y) - g(x)| dy \end{aligned}$$



Even though the $B(x^0, \delta)$ term is **BAD**, one can show that it is very small, $< \epsilon$, therefore we only need to study the second term.

STEP 2:

$$\begin{aligned} \int_{\mathbb{R}^n \setminus B(x^0, \delta)} \Phi(x - y, t) \underbrace{|g(y) - g(x)|}_{\leq C} dy &\leq C \int_{\mathbb{R}^n \setminus B(x^0, \delta)} \Phi(x - y, t) dy \\ &= C \int_{\mathbb{R}^n \setminus B(x^0, \delta)} \Phi(y - x, t) dy \end{aligned}$$

(The last step is because Φ is even, so $\Phi(x - y, t) = \Phi(y - x, t)$)

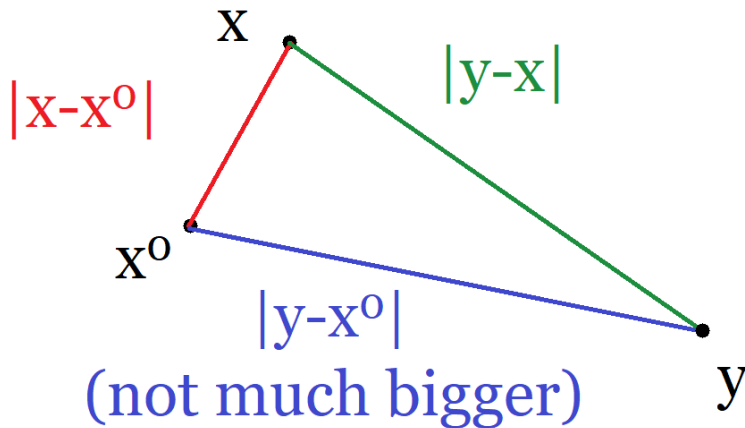
Idea: IF we had $\Phi(y - x^0, t)$ instead of $\Phi(y - x, t)$, then we could do a change of variables which transforms the ball $B(x^0, \delta)$ into $B(0, \text{Blah})$, which is much nicer.

STEP 3: Geometric Lemma

For this, we need a geometric lemma:

Geometric Lemma:

If $|x - x^0| < \frac{\delta}{2}$ and $|y - x^0| > \delta$, then $|y - x^0| < 2|y - x|$



What this is saying is that if x and x^0 are close together, and y is far away from x^0 , then it could happen that $|y - x^0|$ is bigger than $|y - x|$. But this is saying that $|y - x^0|$ not *much* bigger than $|y - x|$, it can never be more than twice as big as $|y - x|$

In other words, it *could* happen that y is further away from x^0 than from x , but y cannot be more than twice as much away from x^0 than from x (which makes sense since x and x^0 are close)

Proof:

$$|y - x^0| \leq |y - x| + \underbrace{|x - x^0|}_{< \frac{\delta}{2}} < |y - x| + \frac{\delta}{2} < |y - x| + \frac{|y - x^0|}{2}$$

Hence, solving for $|y - x^0|$ in the above equation, we get $\frac{1}{2}|y - x^0| < |y - x|$, that is $|y - x^0| < 2|y - x|$ \square

STEP 4: Now going back to $C \int_{\mathbb{R}^n \setminus B(x^0, \delta)} \Phi(y - x, t) dy$.

In particular, since we now know that $|y - x| \geq \frac{1}{2}|y - x^0|$, then

$$\Phi(y - x, t) = \frac{C}{t^{\frac{n}{2}}} e^{-\frac{|y-x|^2}{4t}} \leq \frac{C}{t^{\frac{n}{2}}} e^{-\frac{|y-x^0|^2}{4(4t)}} = \frac{C}{t^{\frac{n}{2}}} e^{-\frac{|y-x^0|^2}{16t}}$$

We obtain:

$$\begin{aligned} C \int_{\mathbb{R}^n \setminus B(x^0, \delta)} \Phi(y - x, t) dy &\leq \frac{C}{t^{\frac{n}{2}}} \int_{\mathbb{R}^n \setminus B(x^0, \delta)} e^{-\frac{|y-x^0|^2}{16t}} \\ &\leq 2^n \pi^{\frac{n}{2}} \int_{\mathbb{R}^n \setminus B(0, \frac{\delta}{\sqrt{t}})} e^{-|z|^2} dz \end{aligned}$$

(Here we used the promised change of variables $z = \frac{y-x^0}{\sqrt{16t}}$)

STEP 5: Finally, we can write the above integral as

$$2^n \pi^{\frac{n}{2}} \int_{\mathbb{R}^n} 1_{\mathbb{R}^n \setminus B(0, \frac{\delta}{\sqrt{t}})} e^{-|z|^2} dz$$

Now since $\frac{\delta}{\sqrt{t}} \rightarrow \infty$ as t goes to 0^+ , the function $1_{\mathbb{R}^n \setminus B(0, \frac{\delta}{\sqrt{t}})}$ converges pointwise to 0, and moreover the integral is dominated by $\int_{\mathbb{R}^n} e^{-|z|^2} dz < \infty$, hence by the dominated convergence theorem,

$$\int_{\mathbb{R}^n \setminus B(0, \frac{\delta}{\sqrt{t}})} e^{-|z|^2} dz \rightarrow 0$$

And can be made $< \epsilon$ if t is small enough.

STEP 6: So, combined with STEP 1, we then get

$$|u(x, t) - g(x^0)| < 2\epsilon \quad \square$$