LECTURE 6: HEAT EQUATION PROPERTIES

Readings:

- Infinite Propagation Speed (section 1 below)
- Section 2.3.1c: Nonhomogeneous Problem (page 49-51)
- Section 2.3.2: Mean-Value Formula (page 52-54)
- Section 2.3.3a: Strong Maximum Principle (up to and including page 56)

Now that we found the fundamental solution of the heat equation and solved the initial-value problem, this week we'd like to explore some properties of the heat equation, like the Mean-Value formula and the Maximum Principle

1. Infinite Propagation Speed

Recall:

One solution of the initial value problem

$$\begin{cases} u_t = \Delta u \\ u(x,0) = g(x) \end{cases}$$

is given by

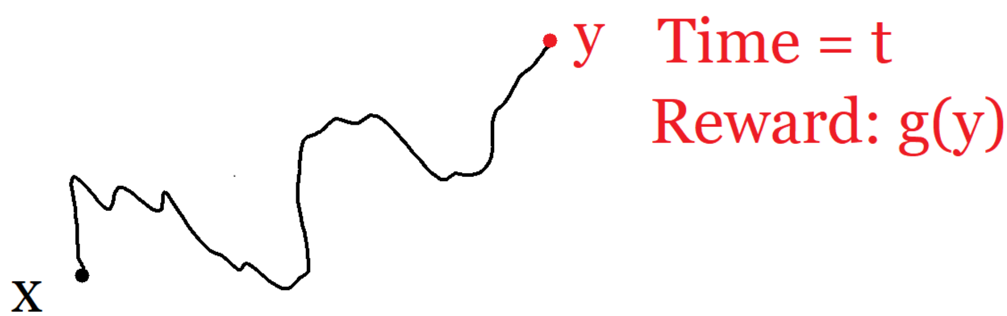
$$u(x,t) = \Phi \star g = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y) dy$$

Date: Monday, May 4, 2020.

Notice: $e^{-\frac{|x-y|^2}{4t}} > 0$, which means that if $g \geq 0$ and $g > 0$ *somewhere*, then $u(x,t) > 0$ for *all* x and t.

Interpretation: This is what's called **infinite propagation speed**. In other words, if an alien light-years away lights a match, then you *immediately* feel the heat of it. This is very different from the wave equations, as we'll see in a couple of weeks.

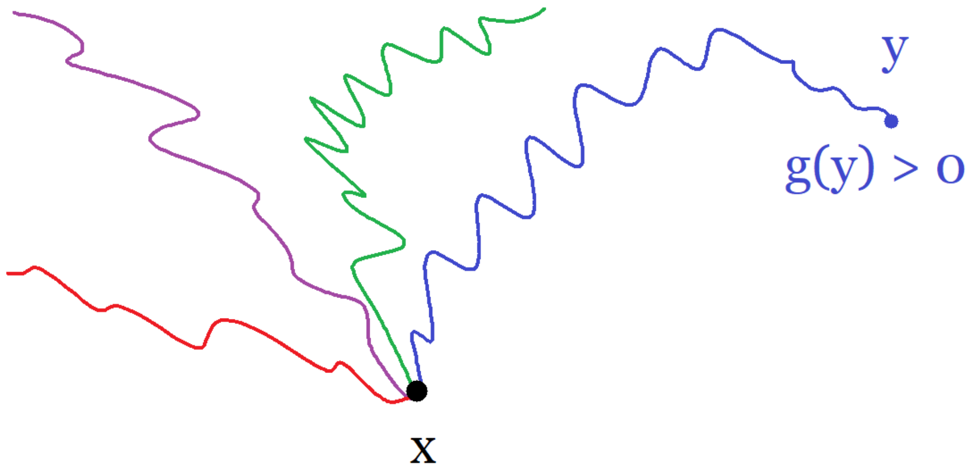
Applications: Remember that you can think of the heat equation in terms of Brownian Motion:



That is, suppose you do Brownian motion at x until I tell you to stop (at a certain time t). Then, your reward is $g(y)$, where y is the point you stop at.

Then if $u(x,t) =$ expected reward starting at x and time t, then we've seen that u solves the heat equation with $u(x,0) = g(x)$.

Now suppose $g > 0$ somewhere, as in the following picture:



Then $u > 0$ for *all* x and *all* t.

In other words, there's always a positive chance of reaching any point at any time t.

Consequence: Brownian Motion is *isotropic* (= the same everywhere)

Careful: This is NOT the same as asking whether, if you start at x, you'll eventually be able to get back to x; that is, whether Brownian motion is recurrent. Interestingly, in two dimensions, the answer is yes (you'll always be able to go back to where you started from), but in 3 dimensions, the answer is no. And in fact there's a famous saying by Kakutani that says "A drunken man will eventually find his way home, but a drunken bird may get lost forever."

2. Inhomogeneous Problem

Reading: Section 2.3.1c: Nonhomogeneous Problem (page 49-51)

Now let's solve the analog Poisson's equation $-\Delta u = f$, but for the heat equation, that is:

Inhomogeneous Problem:

$$\begin{cases} u_t - \Delta u = f(x,t) \\ u(x,0) = 0 \end{cases}$$

Where $f(x,t)$ is a *given* function.

Before, for the initial value problem, we used the convolution with respect to x, that is

$$\Phi \star g = \int_{\mathbb{R}^n} \Phi(x-y,t) g(y) dy$$

But this time we need to use the **FULL** convolution (with respect to x and t), that is:

$$\begin{aligned} u(x,t) = \Phi \star f &= \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s) f(y,s) dy ds \\ &= \int_0^t \int_{\mathbb{R}^n} \Phi(y,s) f(x-y,t-s) dy ds \end{aligned}$$

(Again, think sum $= x$, so $x - y + y = x$ and sum $= t$ so $t - s + s = t$. Also, we integrate from 0 to t since, by convention, $\Phi(x,s) = 0$ for $s < 0$)

Claim:

$u = \Phi \star f$ solves $u_t - \Delta u = f$

Proof-Sketch: (Please read the full proof in the book)

STEP 1: First of all,

$$\Delta u = \Delta\left(\int\int \Phi f\right) = \int\int \Phi(\Delta f)$$

The point is that the Δu part is not a problem

STEP 2: It's the u_t that's a problem, since there are two t here in the integral

$$u(x,t) = \int_0^t \Phi(y,s) f(x-y, t-s)\,dy\,ds$$

Recall: Chain Rule

$$\frac{d}{dt}\left(f(g(t))\right) = f'(g(t))g'(t)$$

Chain Rule for Integrals

$$\frac{d}{dt}\left(\int_0^t g(x, t-s)ds\right) = g(x, t-t) + \int_0^t g_t(x, t-s)ds$$

Note: It *is* just the Chain rule, if you let $F(t) = \int_0^t g(x,s)ds$

Therefore here, by the Chain rule for integrals:

$$
\begin{aligned}
u_t =& \frac{d}{dt}\left(\int_0^t \int_{\mathbb{R}^n} \Phi(y,s) f(x-y,t-s) ds\right) \\
=& \int_{\mathbb{R}^n} \Phi(y,t) f(x-y,t-t) dy + \int_0^t \int_{\mathbb{R}^n} \Phi(y,s) f_t(x-y,t-s) dyds \\
=& \int_{\mathbb{R}^n} \Phi(y,t) f(x-y,0) dy - \int_0^t \int_{\mathbb{R}^n} \Phi(y,s) f_s(x-y,t-s) dyds
\end{aligned}
$$

Here we used $f_t(x-y,t-s) = -f_s(x-y,t-s)$

STEP 3: Therefore:

$$
u_t - \Delta u = \int_0^t \int_{\mathbb{R}^n} \Phi\left(-f_s - \Delta f\right) dyds + \int_{\mathbb{R}^n} \Phi(y,t) f(x-y,0)
$$

The problem is that in the $\int_0^t$ term, Φ has a singularity near 0, so we need to split up the integral as

$$
\int_0^\epsilon \int_{\mathbb{R}^n} + \int_\epsilon^t \int_{\mathbb{R}^n} + \int_{\mathbb{R}^n} \Phi f(0)
$$

Then you show that the first term is small, and for the second term, you show that it's equal to $\int \Phi f - \int \Phi f(0)$, and therefore the above becomes (notice the *MIRACULOUS* cancellation)

$$
\int \Phi f - \cancel{\int \Phi f(0)} + \cancel{\int \Phi f(0)} = \int \Phi f
$$

And you can then show that the latter goes to f as $\epsilon \to 0$ (similar to how we solved Poisson's equation).

Therefore $u_t - \Delta u = f$.

STEP 4: Finally, you get $u(x,0) = 0$ by writing:

$$|u| = \left| \int_0^t \int \Phi f \right| \leq \int_0^t \int \Phi \, |f| \leq C \int_0^t \int \Phi = C \int_0^t 1 = Ct \to 0 \text{ (as } t \to 0)$$

Where we used $\int \Phi = 1$ by definition of the fundamental solution $\square$

3. The Mean-Value Formula

Reading: Section 2.3.2: Mean-Value Formula (page 52-54)

Recall: Mean-Value Formula for Laplace

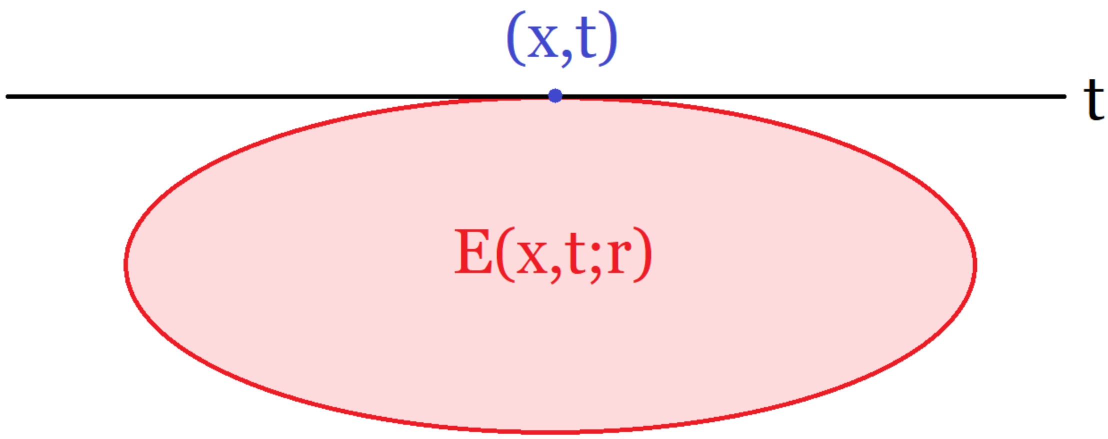
If $\Delta u = 0$, then

$$\frac{\int_{B(x,r)} u(y)dy}{\alpha(n) r^n} = u(x)$$

Now here, the analog of $B(x,r)$ is the heat ball:

Heat Ball

$$E(x,t;r) = \left\{ (y,s) \mid s \leq t, \Phi(x-y, t-s) \geq \frac{1}{r^n} \right\}$$



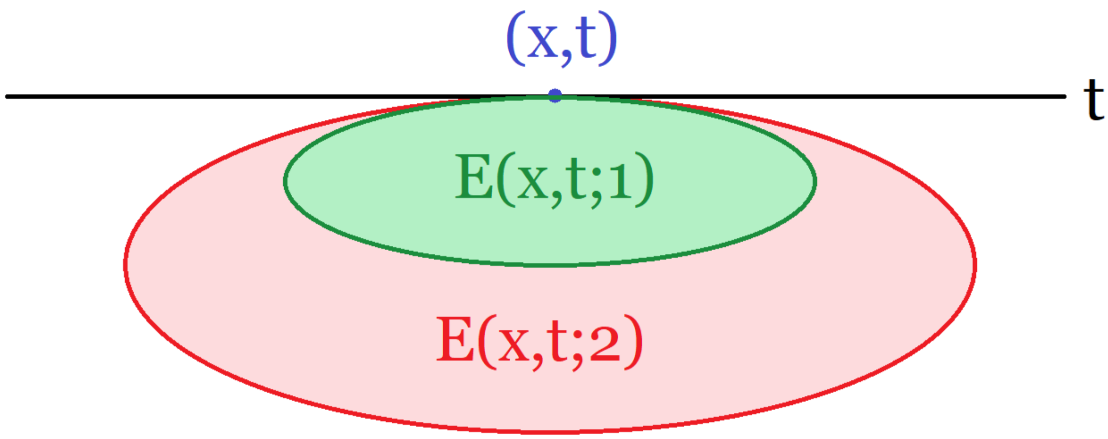
Note: The first condition says that t is on top of the ball.

Note: To convince you that this is really a ball, let's plug in some values of r.

Example: For $r = 1$ this becomes $\Phi(x - y, t - s) \geq 1$

Example: For $r = 2$ this becomes $\Phi(x - y, t - s) \geq \frac{1}{2^n}$. Notice that $\frac{1}{2^n}$ is smaller than 1, so *more* points satisfy $\Phi(x - y, t - s) \geq \frac{1}{2^n}$.

Hence $E(x, t; 2)$ is much bigger than $E(x, t; 1)$, so it is really a ball.



Now, without further ado, here is the mean value formula for the heat equation:

Mean-Value Formula for the Heat Equation

$$u(x,t) = \frac{\int\int_{E(x,t;r)} u(y,s) \frac{|x-y|^2}{(t-s)^2} dyds}{4r^n}$$

Note: Compare this with the mean-value formula for Laplace's equation, in particular the terms in red above. Also, the term $\frac{|x-y|^2}{(t-s)^2}$ is kind of a normalizing factor (a factor that makes this work)

Proof-Sketch: Similar idea to Laplace's equation:

STEP 1: WLOG, after a translation assume $x = 0$ and $t = 0$, and let $E(r) = E(0, 0; r)$. Define

$$\phi(r) = \frac{1}{r^n} \int\int_{E(r)} u(y,s) \frac{|y|^2}{s^2} dyds = \int\int_{E(1)} u(ry, r^2 s) \frac{|y|^2}{s^2} dyds$$

(For the last one, you use the change of variables $y' = \frac{y}{r}$ and $s' = \frac{s}{r^2}$)

STEP 2: Then show $\phi'(r) = 0$

This step is kind of ugly, but basically you write $\phi'(r) = A + B$ and after integrating by parts you show $B =$ JUNK $- A$ and therefore you get:

$$\phi'(r) = A + B = A + \text{ JUNK } - A = \text{ JUNK}$$

STEP 3: Therefore $\phi(r)$ is constant, and so

$$\begin{aligned}
\phi(r) =& \lim_{t\to 0} \phi(t) \\
=& \lim_{t\to 0} \frac{1}{t^n} \int\int_{E(t)} \underbrace{u(y,s)}_{\approx u(0,0)} \frac{|y|^2}{s^2} dyds \\
=& u(0,0) \lim_{t\to 0} \frac{1}{t^n} \int\int_{E(t)} \frac{|y|^2}{s^2} dyds \\
=& u(0,0) \lim_{t\to 0} \int\int_{E(1)} \frac{|y|^2}{s^2} dyds \qquad \text{(Change of variables)} \\
=& u(0,0) \int\int_{E(1)} \frac{|y|^2}{s^2} dyds \\
=& u(0,0)4
\end{aligned}$$

Note: That last integral is **not** trivial. Check out this link if you want to see a proof of it Heat Ball Integral

Therefore

$$\frac{1}{r^n} \int\int_{E(r)} u(y,s) \frac{|y|^2}{s^2} dyds = 4u(0,0)$$

That is:

$$u(0,0) = \frac{1}{4r^2} \int\int_{E(r)} u(y,s) \frac{|y|^2}{s^2} dyds$$

And translating back, we get

$$u(x,t) = \frac{1}{4r^2} \int\int_{E(x,t;r)} u(y,s) \frac{|x-y|^2}{(t-s)^2} dyds \quad \square$$

4. Strong Maximum Principle

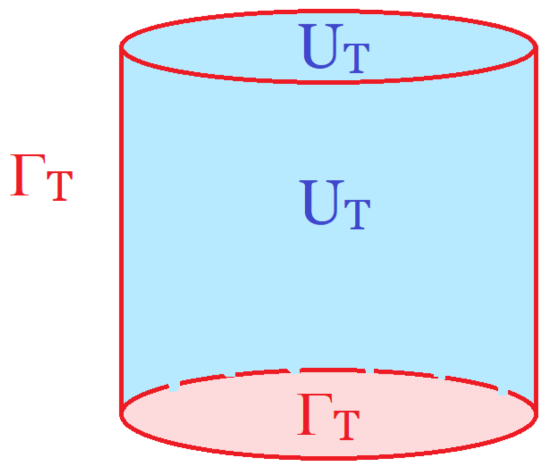
Reading: Section 2.3.3a: Strong Maximum Principle (up to and including page 56)

Same as Laplace, but with a twist

Notation:

$$U_T = U \times (0,T]$$
$$\Gamma_T = \overline{U_T} \backslash U_T$$

Note: Think of Γ_T like a cup: It contains the sides and bottoms, but not the top. And think of U_T as water: It contains the inside and the top.

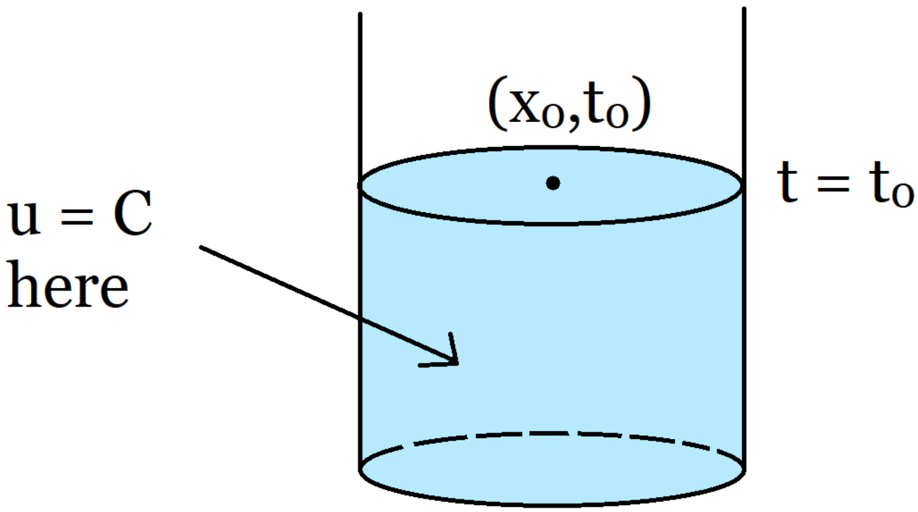


Weak Maximum Principle:

$$\max_{\overline{U}_T} u = \max_{\Gamma_T} u$$

Strong Maximum Principle:

If the max of u is attained at (x_0, t_0) for some $x_0 \in U$, then u is constant in $\overline{U}_{t_0}$



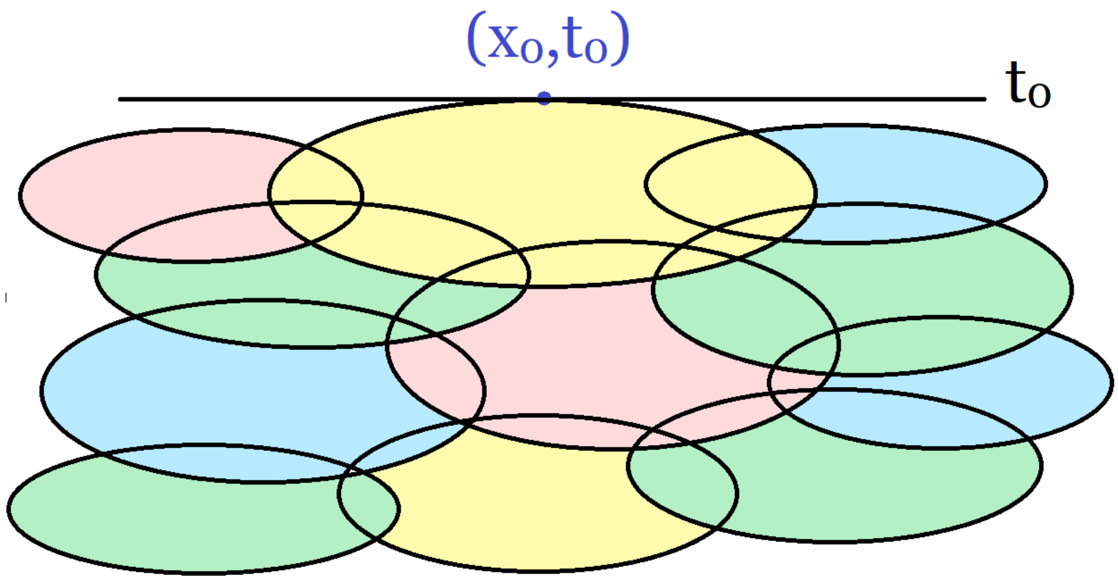
In other words, u is constant for all earlier times. The reason why this is so weird is because for the heat equation, we don't know what the future holds; all that we can say is that u is constant up to now. Maybe tomorrow a meteorite will hit us and it'll get much warmer all of the sudden.

Proof-Sketch: Just a consequence of the mean-value formula. Suppose u has a maximum M at (x_0, t_0), then by the mean-value formula, we have:

$$M = u(x_0, t_0) = \frac{1}{4r^n} \int \int_{E(x,t;r)} u(y, s) \frac{|x_0 - y|^2}{(t_0 - s)^2} dyds$$

But the biggest possible value of the right-hand-side is M, and that's only if $u \equiv M$ on $E(x, t; r)$.

Finally, cover your region with heat balls



Note: Because your heat balls are of the form $s \leq t$, we will never be able to go beyond t_0. That's why you can only cover all of $\overline{U_{t_0}}$ and nothing beyond that.