LECTURE 7: HEAT EQUATION AND ENERGY METHODS

Readings:

- Section 2.3.4: Energy Methods
- Convexity (see notes)
- Section 2.3.3a: Strong Maximum Principle (pages 57-59)

This week we'll discuss more properties of the heat equation, in particular how to apply energy methods to the heat equation. In fact, let's start with energy methods, since they are more fun!

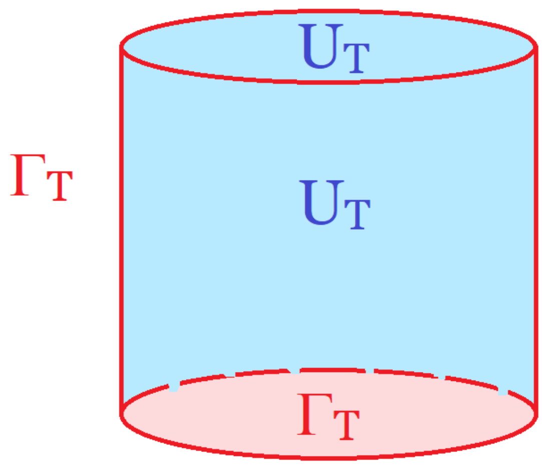
Recall the definition of parabolic boundary and parabolic cylinder from last time:

Notation:

$$U_T = U \times (0, T]$$
$$\Gamma_T = \overline{U_T} \backslash U_T$$

Note: Think of Γ_T like a cup: It contains the sides and bottoms, but not the top. And think of U_T as water: It contains the inside and the top.

Date: Monday, May 11, 2020.



Weak Maximum Principle:

$$\max_{\overline{U_T}} u = \max_{\Gamma_T} u$$

1. Uniqueness

Reading: Section 2.3.4: Energy Methods

Fact:

There is at most one solution of:

$$\begin{cases} u_t - \Delta u = f & \text{in } U_T \\ u = g & \text{on } \Gamma_T \end{cases}$$

We will present two proofs of this fact: One using energy methods, and the other one using the maximum principle above

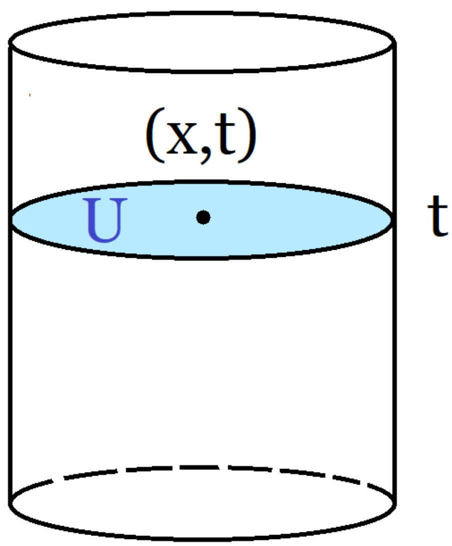
Proof: Suppose u and v are solutions, and let $w = u - v$, then w solves

$$\begin{cases} w_t - \Delta w = 0 & \text{in } U_T \\ u = 0 & \text{on } \Gamma_T \end{cases}$$

Proof using Energy Methods:

Now for fixed t, consider the following energy:

$$E(t) = \int_U (w(x,t))^2 \, dx$$



Then:

$$E'(t) = \int_U 2w\,(w_t) = \int_U 2w\Delta w \overset{IBP}{=} -2\int_U |Dw|^2 \leq 0$$

Therefore $E'(t) \leq 0$, so the energy is decreasing, and hence:

$$(0 \leq) E(t) \leq E(0) = \int_U (w(x,0))^2 \, dx = \int 0 = 0$$

And hence $E(t) = \int w^2 \equiv 0$, which implies $w \equiv 0$, so $u - v \equiv 0$, and hence $u = v$ $\square$.

Maximum Principle Proof: Let $w = u - v$ be as above, then

$$\max_{\overline{U}_T} w = \max_{\Gamma_T} w = \max_{\Gamma_T} 0 = 0$$

So the biggest value of w is 0, hence $w \leq 0$

Similarly

$$\min_{\overline{U}_T} w = \min_{\Gamma_T} w = \min_{\Gamma_T} 0 = 0$$

So the smallest value of w is 0, hence $w \geq 0$

And therefore $w = 0$, so $u - v = 0$, so $u = v$ $\square$

Which method do you like more? Let me know ☺

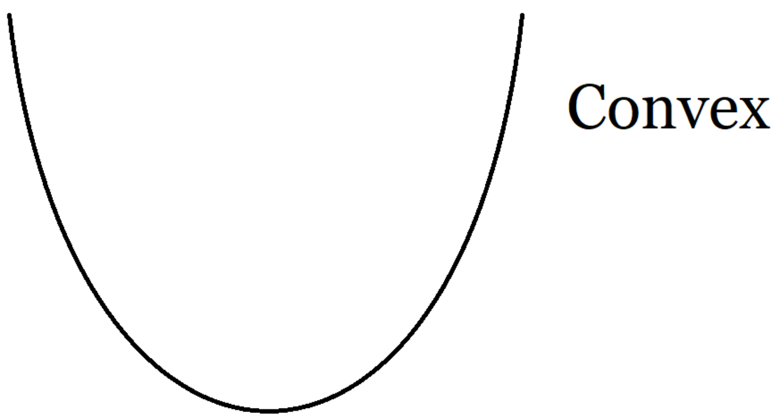
Note: In general, maximum principle methods are good when you want to prove results about u at a point (x,t) like "Show $u(x,t) \leq 2$." Energy methods are good when you want to prove results about $\int u$, like "Show $\int (u(x,t))^2 \, dx < \infty$

2. Convexity

We will do one more fun energy methods exercise, but for this we need to define what it means for a function to be **convex**.

Note: This is *extremely* important if you do any kind of optimization theory.

Intuitively: Convex = Concave up (from calculus), like x^2 or e^x



It turns out there are 3 characterizations of convexity, depending on how many derivatives you have. They are equivalent if f is twice differentiable, although this is not obvious. Here everything is in 1 dimensions:

Definition 1: (2 derivatives)

$f : \mathbb{R} \to \mathbb{R}$ is convex if, for all x,

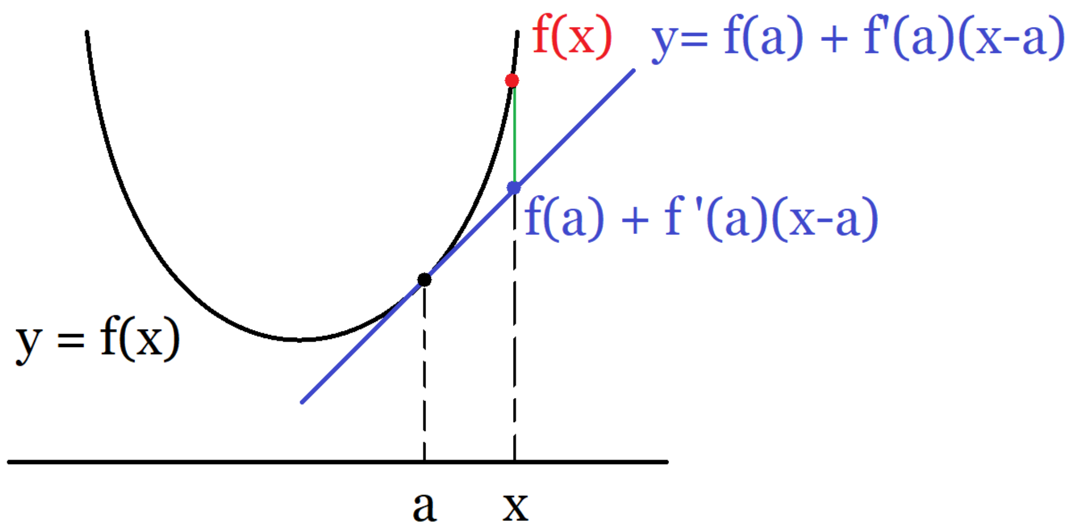
$$f''(x) \geq 0$$

Definition 2: (1 derivative)

f is convex if, for any $a \in \mathbb{R}$ and any $x \in \mathbb{R}$,

$$f(x) \geq f(a) + f'(a)(x-a)$$

Interpretation: f is above any of its tangent lines

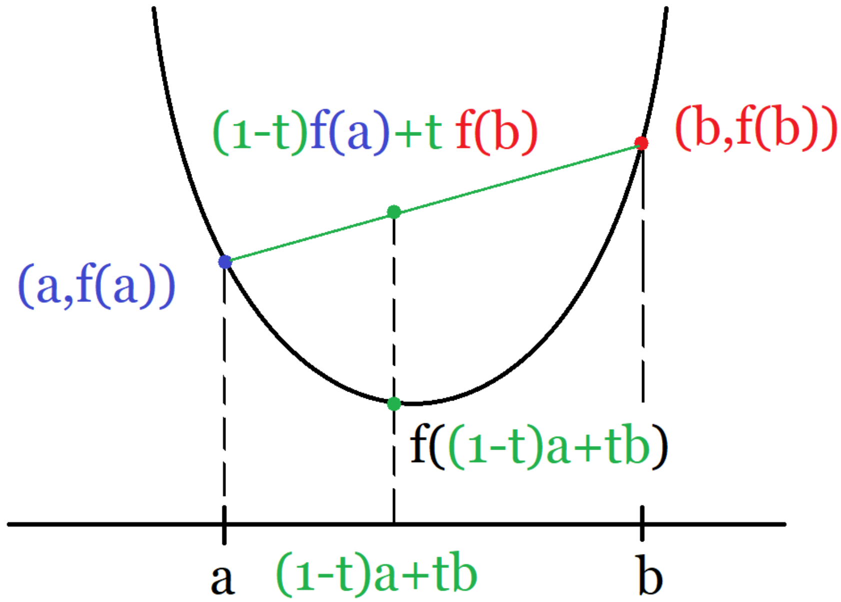


Note: This is extremely useful because sometimes you want to show $f \geq$ something, and convexity tells you precisely that: $f(x) \geq f(a) + f'(a)(x-a)$

Definition 3: (No derivatives)

f is convex if, for any $a, b \in \mathbb{R}$ and any $0 \leq t \leq 1$,

$$f((1-t)a + tb) \leq (1-t)f(a) + tf(b)$$



Interpretation: f is below any chord connecting $(a, f(a))$ and $(b, f(b))$

Note: If the t confuses you, try out this definition with $t = \frac{1}{2}$. Then $(1-t)a + tb$ becomes the midpoint of $[a, b]$ and $(1-t)f(a) + tf(b)$ becomes the midpoint of $f(a)$ and $f(b)$

It's the last characterization that we will use below.

3. Backwards Uniqueness

As an application of energy methods and convexity, the following fact that is not obvious at all. Here T is some terminal time (think $T = 2$ hours or $T = 30$ days)

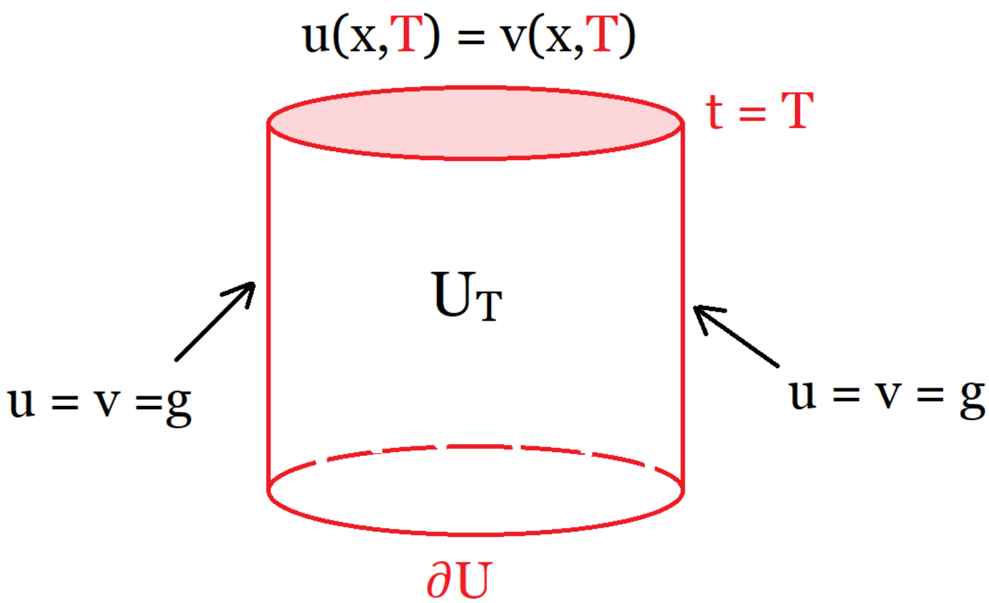
Backwards Uniqueness:

Suppose u and v solve

$$\begin{cases} u_t = \Delta u \text{ in } U_T \\ u = g(x,t) \text{ on } \partial U \end{cases} \qquad \begin{cases} v_t = \Delta v \text{ in } U_T \\ v = g(x,t) \text{ on } \partial U \end{cases}$$

And moreover suppose that $u(x,T) = v(x,T)$

Then $u \equiv v$ in $\overline{U_T}$



In other words: Suppose u and v both solve the heat equation and are equal to g on the lateral part (= sides) of the parabolic cylinder. Then if they're equal at the *terminal* time T, then they must have been equal all along (meaning for all x and for all t up to T).

The reason it's called backwards uniqueness is that if you know the terminal value $u(x,T)$ is the value of u on the sides, then you can figure out what u is everywhere. This is really surprising because the heat equation is generally irreversible, you generally cannot go backwards in time for the heat equation. But this is saying that you *can* go backwards in time, provided you know what u is on the sides!

(The way I picture is that if you put two cakes of the same shape in the oven, in such a way that you know what the temperature is at the sides at all times. Then if the temperature when you take them out is the same, then the temperature of the two must have been the same at all times).

Notice in particular that there is absolutely no assumption on the initial terms $u(x,0)$ and $v(x,0)$; that's what makes this so great!

Proof: This is one of the beautiful moments in math where the proof is more amazing than the result itself:

STEP 1: As usual, let $w = u - v$, then w solves:

$$\begin{cases} w_t = & \Delta w \text{ in } U_T \\ w = & 0 \text{ on } \partial U \end{cases}$$

And moreover $w(x,T)$ for all $x \in U$

STEP 2: Define the energy:

$$E(t) = \int_U (w(x,t))^2 \, dx$$

Then:

$$E'(t) = \int_U 2ww_t = \int_U 2w\Delta w = -2\int_U Dw \cdot Dw = -2\int_U |Dw|^2$$

Moreover:

$$E''(t) = -4\int_U Dw \cdot Dw_t \overset{IBP}{=} 4\int_U (\Delta w)w_t = 4\int_U (\Delta w)(\Delta w) = 4\int_U (\Delta w)^2$$

STEP 3: It turns out that there is a beautiful relationship between E, E', E'':

$$E'(t) = -2\int_U Dw \cdot Dw \overset{IBP}{=} 2\int_U w(\Delta w)$$

Now use one of the most important inequalities in PDE:

Cauchy-Schwarz Inequality:

If f and g are any functions, then

$$\int_U fg \leq \left(\int_U f^2\right)^{\frac{1}{2}} \left(\int_U g^2\right)^{\frac{1}{2}}$$

Therefore:

$$E'(t) = 2\int_U w(\Delta w) \leq 2\left(\int_U w^2\right)^{\frac{1}{2}} \left(\int_U (\Delta w)^2\right)^{\frac{1}{2}}$$

That is:

$$(E'(t))^2 \leq 4\left(\int_U w^2\right)\left(\int_U (\Delta w)^2\right) = \left(\int_U w^2\right)\left(\int_U 4(\Delta w)^2\right) = E(t)E''(t)$$

Therefore:

$$\left(E'(t)\right)^2 \leq E(t)E''(t)$$

STEP 4: Here is where convexity comes in play!

Let

$$F(t) = \ln(E(t))$$

Note: Here I'm oversimplifying things a bit because it's possible that $E(t) = 0$, in which case F is undefined.

Then

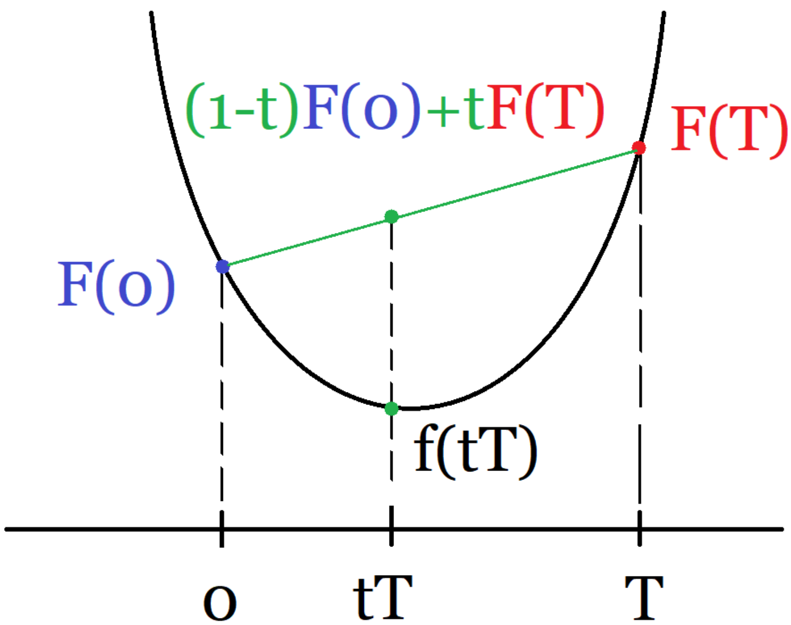
$$F'(t) = \frac{E'(t)}{E(t)}$$

So

$$F''(t) = \frac{E''(t)E(t) - \left(E'(t)\right)^2}{\left(E(t)\right)^2} \geq 0$$

Therefore F is **CONVEX!!!** (by the second derivative definition of convexity)

And therefore, by convexity again (this time the 0 derivative definition of convexity, with chords) with $a = 0$ and $b = T$, we get



$$F(tT) \leq (1-t)F(0) + tF(T)$$

That is:

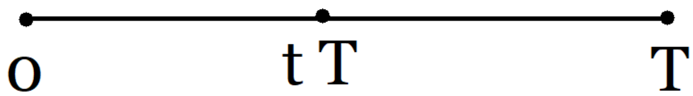
$$\ln(E(tT)) \leq (1-t)\ln(E(0)) + t\ln(E(T))$$

$$e^{\ln(E(tT))} \leq e^{(1-t)\ln(E(0)) + t\ln(E(T))}$$

$$(0 \leq) E(tT) \leq (E(0))^{1-t}(E(T))^{t}$$

However $E(T) = \int_U (w(x,T))^2 = 0$ by assumption on the terminal value, and hence

$$0 \leq E(tT) \leq 0 \Rightarrow E(tT) = 0 \text{ for all } t$$



Therefore $E(t) \equiv 0$ on $[0, T]$, so $\int (w(x,t))^2 = 0$ for all t, so $w \equiv 0$ $\square$

4. The Cauchy Problem

Reading: Section 2.3.3a: Strong Maximum Principle (pages 57-59)

So far we have seen that the (weak) maximum principle is true, at least if your domain U is bounded.

Weak Maximum Principle:

$$\max_{\overline{U}_T} u = \max_{\Gamma_T} u$$

But what if $U = \mathbb{R}^n$? Is the maximum principle still true? It turns out that in general it is not true, but it *is* true if u doesn't grow *too* fast.

Cauchy Maximum Principle:

Suppose u solves:

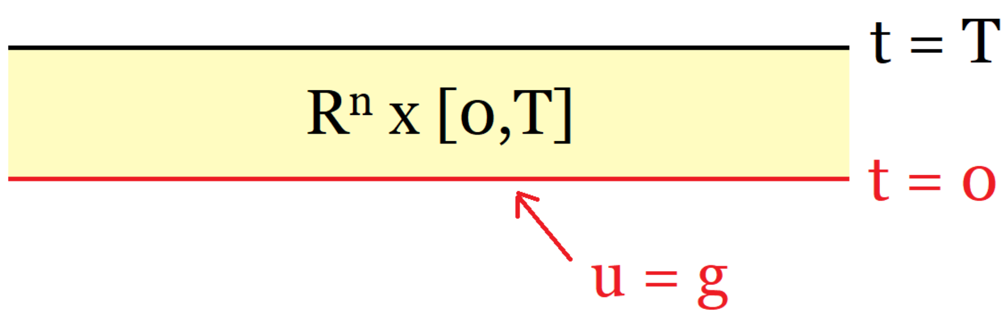
$$\begin{cases} u_t = \Delta u & \text{in } \mathbb{R}^n \times (0,T) \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$$

AND suppose for some $A, a > 0$, we have

$$|u(x,t)| \leq Ae^{a|x|^2}$$

Then:

$$\sup_{\mathbb{R}^n \times [0,T]} u = \sup_{\mathbb{R}^n} g$$



So if u doesn't grow more than square exponentially, then the Cauchy maximum principle is true.

Note: Generally problems on $\mathbb{R}^n$ where you specify $u(x,0)$ are called Cauchy problems.

Corollary: Uniqueness

There is at most one solution of

$$\begin{cases} u_t - \Delta u = f & \text{in } \mathbb{R}^n \times (0,T) \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$$

That satisfies

$$|u(x,t)| \leq Ae^{a|x|^2}$$

Proof: The usual trick where you let u and v be solutions and let $w = u - v$.

However, there are many solutions even of the simple equation

$$\begin{cases} u_t = \Delta u & \text{in } \mathbb{R}^n \times (0,T) \\ u = 0 & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$$

But many of them grow faster than $Ae^{a|x|^2}$ and are hence non-physical. This is very typical in PDE: Usually your equation has many solutions, and you need to rule out solutions that physically make no sense.

Sketch of Proof of the Cauchy Maximum Principle:

This is super hand-wavy but is meant to give you the gist of the proof:

First, introduce a helper function v of the form

$$v = u - \left(Ce^{|x|^2}\right) \text{ term}$$

Where C depends on two parameters ϵ and μ. This seems weird, but more parameters means more freedom.

Then show that on $\overline{U_T}$, we have

$$
\begin{aligned}
v \leq & u - \left(Ce^{|x|^2}\right) \\
\leq & Ae^{a|x|^2} - Ce^{|x|^2} \quad \text{(By assumption)} \\
\leq & \sup g \quad \text{(By choosing your \textit{first} parameter small)}
\end{aligned}
$$

Hence $v \leq \sup g$, and now choose your other parameter small to get $u \leq \sup g$ $\square$

Congratulations! We are now done with the heat equation, and in the next two lectures we'll talk about our third PDE, the Wave Equation!