LECTURE 8: THE WAVE EQUATION

Readings:

- Section 2.1: Transport Equation
- The Wave Equation (pages 65-66)
- Section 2.4.1a: D'Alembert's Formula
- Section 4 of the Lecture Notes: Some consequences
- Section 2.4.1b: Spherical Means

Welcome to the final equation of this course: The Wave Equation

Wave Equation:

$$u_{tt} = \Delta u$$

Compare this with the heat equation $u_t = \Delta u$. Even though they look similar, they actually have different properties!

1. The Transport Equation

Reading: Section 2.1: The Transport Equation

Video: Transport Equation

Date: Monday, May 18, 2020.

Let's first solve a related PDE that will be useful in our solution of the wave equation.

Transport Equation:

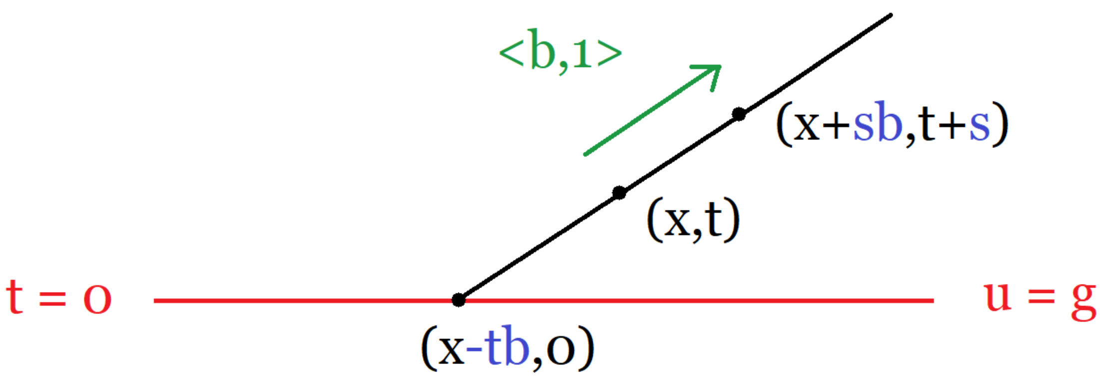
$$\begin{cases} u_t + b \cdot Du = 0 \times \text{ in } \mathbb{R}^n \times (0, \infty) \\ u(x,0) = g(x) \end{cases}$$

Example: In 2 dimensions with $b = (3, -2)$, this becomes

$$u_t + 3u_{x_1} - 2u_{x_2} = 0$$

It turns out this is fairly easy to solve: First of all, the equation $u_t + b \cdot Du = 0$ is suggesting that u is constant on lines directed by $\langle b, 1 \rangle$, which are parametrized by $(x + sb, t + s)$.

Therefore, if you let $z(s) = u(x + sb, t + s)$, then



$$z'(s) = u_{x_1} b_1 + \cdots + u_{x_n} b_n + u_t = u_t + b \cdot Du = 0$$

Therefore $z(s)$ is constant on lines, and hence in particular we get

$$\begin{aligned} z(0) =& z(-t) \\ \Rightarrow u(x+0b, t+0) =& u(x-tb, t-t) \\ \Rightarrow u(x,t) =& u(x-tb, 0) \\ \Rightarrow u(x,t) =& g(x-tb) \end{aligned}$$

Transport Equation:

The solution of the following PDE is

$$\begin{cases} u_t + b \cdot Du = 0 \\ u(x,0) = g(x) \end{cases}$$

$$u(x,t) = g(x-tb)$$

Similarly, we get:

Inhomogeneous version:

The solution of the following PDE is

$$\begin{cases} u_t + b \cdot Du = f(x,t) \\ u(x,0) = g(x) \end{cases}$$

$$u(x,t) = g(x-tb) + \int_0^t f(x+(s-t)b, s)\,ds$$

The proof is the same, except here we don't get $z' = 0$, but $z' = f$ (and so $z = \int f$)

2. The Wave Equation

Reading: Section 2.4: The Wave Equation (pages 65-66)

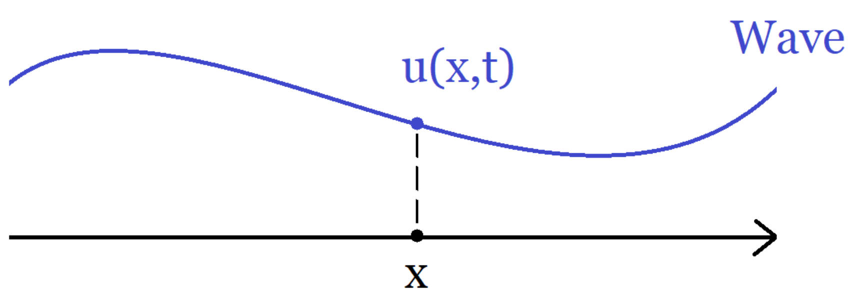
Wave Equation:

$$u_{tt} = \Delta u$$

Derivation: Similar to Laplace's equation or the heat equation, except here you start with the identity $F = ma$ (Force = mass times acceleration)

Applications: The applications of the wave equation depend on the dimension:

(1) (1 dimension) Models a vibrating string: $u(x,t)$ is the height of the string at position x and time t



Also used to model sound waves and light waves

(2) (2 dimensions) Models *water waves*. For example, the wave equation models the water ripples caused by throwing a rock at a pond.

Also used to model a vibrating drum.

(3) (3 dimensions) Models vibrating solids, think like an elastic ball that vibrates

3. D'Alembert's Formula ($n = 1$)

Reading: Section 2.4.1a: D'Alembert's Formula

Video: D'Alembert's Formula

Although Laplace's Equation and the Heat Equation were similar, the Wave equation is *very* different. It not only has different properties, but the derivation is also different.

What makes this even more interesting is that the derivation is different depending on the dimension: We will first do the $1-$dimensional case, then (next time) the $3-$dimensional case, and the $2-$dimensional case.

Goal: $(n = 1)$

Solve:

$$\begin{cases} u_{tt} = u_{xx} \\ u(x,0) = g(x) \\ u_t(x,0) = h(x) \end{cases}$$

(Vibrating string with initial position $g(x)$ and initial velocity $h(x)$)

STEP 1: Clever Observation: We can write $u_{tt} - u_{xx} = 0$ as

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) \underbrace{\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) u}_{v} = 0$$

In particular, if you let $v = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) u = u_t - u_x$, then the above becomes

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) v = 0 \Rightarrow v_t + v_x = 0 \qquad \text{TRANSPORT EQUATION!}$$

Moreover:

$$v(x,0) = u_t(x,0) - u_x(x,0) = h(x) - (g(x))_x = h(x) - g'(x)$$

STEP 2: Therefore we need to solve

$$\begin{cases} v_t + v_x = 0 \\ v(x,0) = h(x) - g'(x) \end{cases}$$

(Transport equation with $b = 1$), which gives:

$$v(x,t) = h(x - tb) - g'(x - tb) = h(x - t) - g'(x - t)$$

STEP 3: Solve for u using $v = u_t - u_x$, that is:

$$\begin{cases} u_t - u_x = & v = \underbrace{h(x-t) - g'(x-t)}_{f(x,t)} \\ u(x,0) = & g(x) \end{cases}$$

(Inhomogeneous transport equation with $b = -1$ and $f(x,t) = h(x - t) - g'(x - t)$), which gives:

$$\begin{aligned} u(x,t) =& g(x - tb) + \int_0^t f(x + (s-t)b, s)\,ds \\ =& g(x+t) + \int_0^t f(x + t - s, s)\,ds \\ =& g(x+t) + \int_0^t h(x + t - s - s) - g'(x + t - s - s)\,ds \qquad \text{(Using def of } f\text{)} \\ =& g(x+t) + \int_0^t h(x + t - 2s) - g'(x + t - 2s)\,ds \end{aligned}$$

$$
\begin{aligned}
=&g(x+t)+\int_{x+t-2(0)}^{x+t-2t} h(s')-g'(s')\left(-\frac{1}{2}ds'\right)\\
&\text{(Change of vars } s'=x+t-2s)\\
=&g(x+t)-\frac{1}{2}\int_{x+t}^{x-t} h(s)-g'(s)ds\\
=&g(x+t)+\frac{1}{2}\int_{x-t}^{x+t} h(s)-g'(s)ds\\
=&g(x+t)+\frac{1}{2}\int_{x-t}^{x+t} h(s)ds-\frac{1}{2}\int_{x-t}^{x+t} g'(s)ds\\
=&g(x+t)+\frac{1}{2}\int_{x-t}^{x+t} h(s)ds-\frac{1}{2}g(x+t)+\frac{1}{2}g(x-t)\\
=&\frac{1}{2}\left(g(x-t)+g(x+t)\right)+\int_{x-t}^{x+t} h(s)ds
\end{aligned}
$$

Which, last but not least, gives the celebrated:

D'Alembert's Formula

The solution of the wave equation in 1 dimensions is

$$
\begin{cases}
u_{tt}= & u_{xx}\\
u(x,0)= & g(x)\\
u_t(x,0)= & h(x)
\end{cases}
$$

$$
u(x,t)=\frac{1}{2}\left(g(x-t)+g(x+t)\right)+\frac{1}{2}\int_{x-t}^{x+t} h(s)ds
$$

4. Some consequences

Let's look at

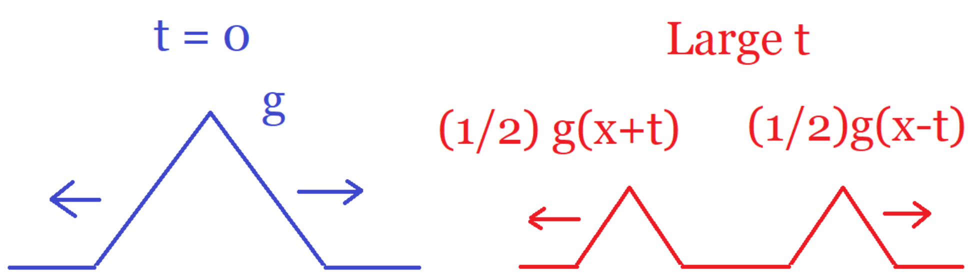
$$u(x,t) = \frac{1}{2}\left(g(x-t) + g(x+t)\right) + \frac{1}{2}\int_{x-t}^{x+t} h(s)ds$$

a bit more.

(1) If $h \equiv 0$, then we get

$$u(x,t) = \frac{1}{2}\left(g(x+t) + g(x-t)\right)$$

Which means that, if there's no initial velocity, the initial wave splits up into two half-waves, one moving to the right and the other one moving to the left.



Note: Check out the following really cool web applet that allows you to simulate solutions of the wave equation by specifying g and h: Wave Equation Simulation

(2) Note that $u(x,t)$ depends **only** on the values of g and h on $[x-t, x+t]$. Values of g and h **outside** of $[x-t, x+t]$ don't affect u at all! This interval is sometimes called the **domain of dependence**. Think of the domain of dependence as a kind of

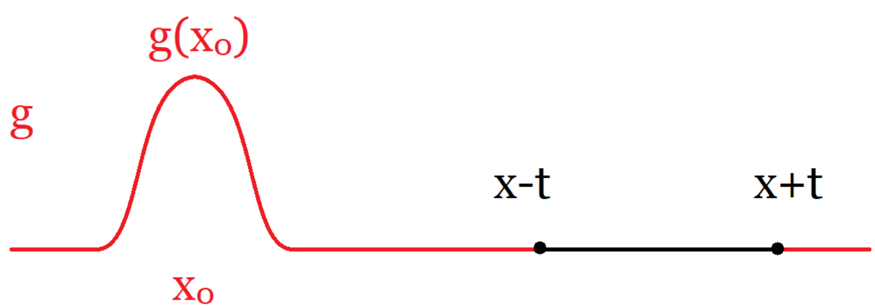
a bunker or safe haven. As long as you're inside of the bunker, nothing in the outside world will affect you.

(3)

Corollary:

The wave equation has **finite** speed of propagation

More precisely, $g(x_0) > 0$ for some x_0 but $g \equiv 0$ inside $[x-t, x+t]$, then $u(x,t) = 0$



This is **very** different from the heat equation, where, as we have seen, if $g(x_0) > 0$ *somewhere*, then $u(x,t) > 0$ everywhere !

Analogy: If an alien (lightyears) away lights a match, then you *immediately* feel the effect of the heat. But if that alien makes a sound, then it will take some time until you heat it (for t so large until x_0 is in $[x-t, x+t]$)

(4) There is no maximum principle for the wave equation; in general $\max u(x,t) \neq \max g$. In other words, your wave $u(x,t)$ could become *bigger* than your initial wave $g(x)$ (think for instance

what happens during resonance).

Or, for example, take $g \equiv 0$ and $h > 0$, then $u(x,t) > 0$ but $\max g \equiv 0$

(5) **Smoothness:** Usually u is not infinitely differentiable. u is generally as smooth as g, and 1 degree smoother than h.

For example, if $g(x) = |x|$ (not differentiable) and $h \equiv 0$, then $u(x,t) = \frac{1}{2}\left(|x-t| + |x+t|\right)$, which is also not differentiable

(6) **Uniqueness:** Generally yes, but need to do it with energy methods since there's no maximum principle

(7) **Reflection Method:** (Optional) If you want to solve the wave equation on the half-line, where this time $x > 0$ (instead of $x \in \mathbb{R}$) then you can use a reflection method. See page 69 of the book, or this video: Reflection of Waves, or pages 3-9 of the following lecture notes Reflection Method. The physical phenomenon is quite interesting, where your wave just *reflects* off a wall. Feel free to check it out

5. Euler-Poisson-Darboux Equation

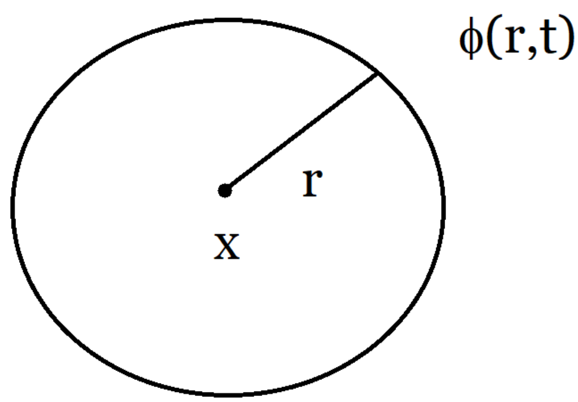
Reading: Section 2.4.1b: Spherical Means

Of course, you may wonder: Is there a mean-value formula for the wave equation? Well yes, but actually no! There isn't a mean-value *formula* here, but actually a mean-value *PDE* called the Euler-Poisson-Darboux equation! This will actually help us next time to solve the wave equation in 3 dimensions

(Carefully note: If a theorem is named after a mathematician (like Fermat's Last Theorem), then it's important. Here it's named after **THREE** mathematician, so it's **VERY** important)

Fix x and let

$$\phi(r,t) = ⨍_{\partial B(x,r)} u(y,t)\,dS(y)$$



Note: Technically, ϕ should also depend on x, but here x will be constant throughout.

Claim:

ϕ solves the following PDE, called the **Euler-Poisson-Darboux Equation**:

$$\phi_{tt} - \phi_{rr} - \left(\frac{n-1}{r}\right)\phi_r = 0$$

With

$$\phi(r,0) = \fint_{\partial B(x,r)} g(y)dS(y) =: G(r)$$

$$\phi_t(r,0) = \fint_{\partial B(x,r)} h(y)dS(y) =: H(y)$$

Note: Compare this to back in section 2.2 when we tried to find the fundamental solution of Laplace's equation, then we found an expression of the form $w'' + \left(\frac{n-1}{r}\right)w'$. In fact, the $\phi_{rr} + \left(\frac{n-1}{r}\right)\phi_r$ term is the radial part of Laplace's equation in polar coordinates, so the above is a sort of a wave equation (and we'll be able to transform it to an actual wave equation next time).

Proof: Similar to the derivation of Laplace's mean value formula!

Note: The initial conditions $\phi(r,0) = G(r)$ and $\phi_t(r,0) = H(r)$ are easy to check from the definition, so let's just focus on the PDE.

STEP 1: Just like for Laplace's equation, let's change variables:

$$
\begin{aligned}
\phi =& \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B(x,r)} u(y,t)dS(y) \\
=& \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B(0,1)} u(x+rz,t)r^{n-1}dS(z) \\
& \text{(Here we used } z = \frac{y-x}{r}\text{)} \\
\phi =& \frac{1}{n\alpha(n)} \int_{\partial B(0,1)} u(x+rz,t)dS(z)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\phi_r =& \frac{1}{n\alpha(n)} \int_{\partial B(0,1)} Du(x+rz,t)\dot{z}dS(z) \\
=& \frac{1}{n\alpha(n)} \int_{\partial B(x,r)} Du(y,t) \cdot \left(\frac{y-x}{r}\right)\left(\frac{1}{r^{n-1}}\right) dS(y) \\
& \text{(Here we used } y = x + rz\text{)} \\
=& \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B(x,r)} \left(\frac{\partial u}{\partial \nu}\right) dS(z) \\
=& \frac{1}{n\alpha(n)r^{n-1}} \int_{B(x,r)} \Delta u dy \\
=& \frac{1}{n\alpha(n)r^{n-1}} \int_{B(x,r)} u_{tt} dy \\
& \text{(By our PDE)}
\end{aligned}
$$

STEP 2: Therefore, we get:

$$
\begin{aligned}
\phi_r =& \frac{1}{n\alpha(n)r^{n-1}} \int_{B(x,r)} u_{tt} dy \\
r^{n-1}\phi_r =& \frac{1}{n\alpha(n)} \int_{B(x,r)} u_{tt} dy \\
\left(r^{n-1}\phi_r\right)_r =& \frac{1}{n\alpha(n)} \left(\int_{B(x,r)} u_{tt} dy \right) \\
=& \frac{1}{n\alpha(n)} \left(\int_0^r \int_{\partial B(x,s) u_{tt} dS(y)} dr \right)_r \\
=& \frac{1}{n\alpha(n)} \int_{\partial B(x,r)} u_{tt} dS(y) \\
=& r^{n-1} \left(\frac{\int_{\partial B(x,r)} u_{tt} dS(y)}{n\alpha(n)r^{n-1}} \right) \\
=& r^{n-1} \fint_{\partial B(x,r)} u_{tt} dS(y) \\
=& r^{n-1} \left(\fint_{\partial B(x,r)} u dS(y) \right)_{tt} \\
=& r^{n-1}\phi_{tt}
\end{aligned}
$$

STEP 3: Hence, we get

$$
\begin{aligned}
\left(r^{n-1}\phi_r\right)_r =& r^{n-1}\phi_{tt} \\
(n-1)r^{n-2}\phi_r + r^{n-1}\phi_{rr} =& r^{n-1}\phi_{tt} \\
(n-1)\phi_r + r\phi_{rr} =& r\phi_{tt} \\
\phi_{tt} = \left(\frac{n-1}{r}\right)\phi_r + \phi_{rr}
\end{aligned}
$$

And therefore, we obtain

$$\phi_{tt} = \phi_{rr} + \left(\frac{n-1}{r}\right)\phi_r \quad \square$$

Note: Next time we'll convert it into an actual wave equation (at least in 3 dimensions).