## LECTURE 9: WAVE EQUATION AND ENERGY METHODS

## Readings:

- Section 2.4.3: Energy Methods
- Section 2.4.1c: Solution for $n=3$ (pages 71-72)
- (Optional) Solution for $n=2$ (pages 73-74)
- (Optional) Section 2.4.2: Nonhomogeneous Problem

Welcome to the second (and final) part of our wave equation adventure! Today is all about energy methods, as well as solving the wave equation in 3 dimensions.

## 1. UniqUENESS

Reading: Section 2.4.3: Energy Methods
Let's first show that the solutions to the wave equation are unique. Since there is no maximum principle, we have to resort to energy methods to show this.

$$
\begin{aligned}
& \text { Recall: } \\
& \qquad \begin{array}{c}
U_{T}=U \times(0, T] \\
\Gamma_{T}=\overline{U_{T}} \backslash U_{T}
\end{array}
\end{aligned}
$$

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Note: $\Gamma_{T}$ is like a cup: It contains the sides and bottoms, but not the top. And $U_{T}$ is like water: It contains the inside and the top.

## Uniqueness:

There is at most one solution of

$$
\begin{cases}u_{t t}-\Delta u=f(x, t) & \text { in } U_{T} \\ u(x, t)=g(x, t) & \text { on } \Gamma_{T} \\ u_{t}(x, 0)=h(x) & \end{cases}
$$

To clarify, this means that $u$ solves the (inhomogeneous) wave equation inside the cup, $u$ is $g$ on the sides and the bottom, and the initial velocity (at the bottom) is $h$

## Proof:

STEP 1: Suppose $u$ and $v$ are solutions, and let $w=u-v$, then $w$ solves

$$
\begin{cases}w_{t t}-\Delta w=0 & \text { in } U_{T} \\ w=0 & \text { on } \Gamma_{T} \\ w_{t}(x, 0)=0 & \end{cases}
$$

STEP 2: For each $t$, consider the following energy

$$
E(t)=\frac{1}{2} \int_{U}\left(w_{t}(x, t)\right)^{2}+|D w(x, t)|^{2} d x
$$

(You formally get this energy by multiplying the PDE by $w_{t}$ and integrating by parts; or think of this as the kinetic energy $\frac{1}{2}$ mass $\times$ speed $^{2}$ )

STEP 3: Then

$$
\begin{aligned}
& E^{\prime}(t)=\int_{U} w_{t} w_{t t}+D w \cdot D w_{t} \\
& \stackrel{I B P}{=} \int_{U} w_{t} w_{t t}-(\Delta w) w_{t} \\
&=\int_{U} w_{t}(\underbrace{w_{t t}-\Delta w}_{0}) \\
&=0
\end{aligned}
$$

STEP 4: Therefore $E$ is constant, and therefore

$$
E(t)=E(0)=\int_{U} \underbrace{\left(w_{t}(x, 0)\right)^{2}}_{0}+\underbrace{|D w(x, 0)|^{2}}_{0}=0
$$

Note: The last term is 0 because $w(x, 0)$ for all $x$ (since $w=0$ on $\Gamma_{T}$ ) and therefore, differentiating this with respect to $x$, we get

$$
D w(x, 0)=0
$$

Therefore

$$
E(t)=\frac{1}{2} \int_{U}\left(w_{t}(x, t)\right)^{2}+|D w(x, t)|^{2} d x \equiv 0
$$

Which can only happen if $w_{t} \equiv 0$ and $D w \equiv 0$, which means $w \equiv C$, but since $w(x, 0)=0$, we get $C=0$ and hence $w \equiv 0$, so $u \equiv v$

## 2. Domain of Dependence

This section is also called What happens in a Cone, stays in a cone
Recall: In 1 dimensions, the wave equation has finite speed of propagation, meaning that $u(x, t)$ only depends on initial values inside the interval $(x-t, x+t)$. In particular, any initial condition outside that interval doesn't affect $u(x, t)$ at all

Analogy: If an alien light years away makes a sound, then it'll take some time for you to hear it


Here we would like to show that this phenomenon is true in any dimensions, except here the analog of $(x-t, x+t)$ is a cone.

## Definition:

The wave cone with apex $\left(x_{0}, t_{0}\right)$ is

$$
K\left(x_{0}, t_{0}\right)=\left\{(x, t)\left|0 \leq t \leq t_{0},\left|x-x_{0}\right| \leq t_{0}-t\right\}\right.
$$

To convince you that $K\left(x_{0}, t_{0}\right)$ is a cone, let's work out a specific example

## Example:

Draw $K\left(x_{0}, 2\right)$ (so $t_{0}=2$ )

First of all, by definition, we have $t \leq 2=t_{0}$
If $t=2$, then $\left|x-x_{0}\right| \leq t_{0}-t=2-2=0$, so $\left|x-x_{0}\right| \leq 0$ is the disk centered at radius 0 , hence just the point $x_{0}$ (in the $t=2$ plane)

If $t=1$, then $\left|x-x_{0}\right| \leq t_{0}-t=2-1=1$, so $\left|x-x_{0}\right| \leq 1$ is the disk centered at $x_{0}$ and radius 1 (in the $t=1$ plane)

Finally, if $t=0$, then $\left|x-x_{0}\right| \leq t_{0}-t=2-0=2$, which is the disk centered at $x_{0}$ and radius 2 (in the $t=0$ plane)


In other words, the horizontal slices are just disks that get bigger and bigger the smaller $t$ is, so $K\left(x_{0}, t_{0}\right)$ is really a cone.


## Claim:

Suppose $u$ solves $u_{t t}=\Delta u$
Moreover, suppose $u=u_{t}=0$ on $B\left(x_{0}, t_{0}\right) \times\{t=0\}$
Then $u \equiv 0$ inside $K\left(x_{0}, t_{0}\right)$


In other words, even if $u$ is CRAZY outside of $B\left(x_{0}, t_{0}\right) \times\{t=0\}$, as long as $u=u_{t}=0$ inside $B\left(x_{0}, t_{0}\right) \times\{t=0\}$, $u$ will still be 0 inside the whole cone $K\left(x_{0}, t_{0}\right)$. In other words anything that happens outside of $B\left(x_{0}, t_{0}\right) \times\{t=0\}$ has no effect on $K\left(x_{0}, t_{0}\right)$.
(Think of the cone as a safe zone, inside you're safe from any disturbances outside the cone)

## Proof:

STEP 1: Define the local energy

$$
e(t)=\frac{1}{2} \int_{B\left(x_{0}, t_{0}-t\right)}\left(u_{t}(x, t)\right)^{2}+|D u(x, t)|^{2} d x
$$



Careful: Since the region itself depends on $t$, we need to differentiate the region as well. Luckily, using the polar coordinate formula, one can show that

$$
\frac{d}{d t} \int_{B(x, t)} \mathrm{BLAH}=\int_{\partial B(x, t)} \mathrm{BLAH}
$$

STEP 2: Therefore, using the chain rule, we get

$$
\begin{aligned}
E^{\prime}(t)= & \int_{B\left(x_{0}, t_{0}-t\right)} u_{t} u_{t t}+D u \cdot D u_{t}-\frac{1}{2} \int_{\partial B\left(x_{0}, t_{0}-t\right)}\left(u_{t}\right)^{2}+|D u|^{2} d S(y) \\
\stackrel{I B P}{=} & \int_{B\left(x_{0}, t_{0}-t\right)} u_{t} u_{t t}+\int_{\partial B\left(x_{0}, t_{0}-t\right)}\left(\frac{\partial u}{\partial \nu}\right) u_{t}-\int_{B\left(x_{0}, t_{0}-t\right)}(\Delta u) u_{t} \\
& -\int_{\partial B\left(x_{0}, t_{0}-t\right)}\left(u_{t}\right)^{2}+|D u|^{2} d S(y) \\
= & \int_{B\left(x_{0}, t_{0}-t\right)} u_{t} \underbrace{\left(u_{t t}-\Delta u\right)}_{0}+\int_{\partial B\left(x_{0}, t_{0}-t\right)}\left(\frac{\partial u}{\partial \nu}\right) u_{t}-\frac{1}{2}\left(u_{t}\right)^{2}-\frac{1}{2}|D u|^{2} d S(y) \\
= & \int_{\partial B\left(x_{0}, t_{0}-t\right)}\left(\frac{\partial u}{\partial \nu}\right) u_{t}-\frac{1}{2}\left(u_{t}\right)^{2}-\frac{1}{2}|D u|^{2} d S(y)
\end{aligned}
$$

STEP 3: However, using
(1) The definition of $\frac{\partial u}{\partial \nu}$
(2) The Cauchy-Schwarz inequality $|a \cdot b| \leq|a||b|$
(3) The fact that $|\nu|=1$ (since $\nu$ is the unit normal vector)
(4) Cauchy's inequality $a b \leq \frac{1}{2} a^{2}+\frac{1}{2} b^{2}$
we get:

$$
\begin{aligned}
\frac{\partial u}{\partial \nu} u_{t} & \leq\left|\frac{\partial u}{\partial \nu} u_{t}\right| \\
& =\left|\frac{\partial u}{\partial \nu}\right|\left|u_{t}\right| \\
& \stackrel{(1)}{=}|D u \cdot \nu|\left|u_{t}\right| \\
& \stackrel{(2)}{\leq}|D u| \underbrace{|\nu|}_{=1}\left|u_{t}\right| \\
& \stackrel{(3)}{=}|D u|\left|u_{t}\right| \\
& \stackrel{(4)}{\leq} \frac{1}{2}|D u|^{2}+\frac{1}{2}\left|u_{t}\right|^{2}
\end{aligned}
$$

And therefore

$$
\left(\frac{\partial u}{\partial \nu} u_{t}\right)-\frac{1}{2}|D u|^{2}-\frac{1}{2}\left|u_{t}\right|^{2} \leq 0
$$

STEP 4: But this implies

$$
e^{\prime}(t)=\int_{\partial B\left(x_{0}, t_{0}-t\right)}\left(\frac{\partial u}{\partial \nu}\right) u_{t}-\frac{1}{2}\left(u_{t}\right)^{2}-\frac{1}{2}|D u|^{2} d S(y) \leq 0
$$

Therefore $e(t)$ is decreasing, and hence

$$
e(t) \leq e(0)=\frac{1}{2} \int_{B\left(x_{0}, t_{0}\right)} \underbrace{\left(u_{t}(x, 0)\right)^{2}}_{0}+\underbrace{|D u(x, 0)|^{2}}_{0} d x=0
$$

(Remember $u=0$ and $u_{t}=0$ on $B\left(x_{0}, t_{0}\right) \times\{t=0\}$ by assumption)
STEP 5: Therefore we get $e(t) \equiv 0$, and so

$$
e(t)=\frac{1}{2} \int_{B\left(x_{0}, t_{0}-t\right)}\left(u_{t}(x, t)\right)^{2}+|D u(x, t)|^{2} d x \equiv 0
$$

Which implies that for all $x$ and $t$, we have $u_{t} \equiv 0$ and $D u \equiv 0$, and so $u$ is constant

But then, since $u(x, 0)=0$ on $B\left(x_{0}, t_{0}\right)$ (bottom of the cone) we then get $u \equiv 0$

## 3. Kirchoff's Formula

Reading: Section 2.4.1c: Solution for $n=3$ (pages 71-72)
After all this energy method excitement, let's go back to solving the wave equation.

Last time: Found the solution of the wave equation in 1 dimensions, and our goal was to find a solution in higher dimensions (here in $n=3$ dimensions).

We also found a mean-value PDE for the wave equation:
Fix $x$ and let

## Fact:

If $u$ solves

$$
\left\{\begin{array}{l}
u_{t t}-\Delta u=0 \\
u(x, 0)=g(x) \\
u_{t}(x, 0)=h(x)
\end{array}\right.
$$

And you let

$$
\begin{aligned}
\phi(r, t) & =f_{\partial B(x, r)} u(y, t) d S(y) \\
G(r) & =: f_{\partial B(x, r)} g(y) d S(y) \\
H(r) & =: f_{\partial B(x, r)} h(y) d S(y)
\end{aligned}
$$

Then $\phi$ solves the Euler-Poisson Darboux Equation:

$$
\left\{\begin{array}{l}
\phi_{t t}-\phi_{r r}-\left(\frac{n-1}{r}\right) \phi_{r}=0 \\
\phi(r, 0)=G(r) \\
\phi_{t}(r, 0)=H(y)
\end{array}\right.
$$

MIRACLE: In 3 dimensions $(n=3)$ we can actually transform this into a 1 -dimensional wave equation!

## Claim:

Given $\phi, G, H$ as above, let

$$
\left\{\begin{array}{l}
\tilde{u}=: r \phi \\
\tilde{G}=: r G \\
\tilde{H}=: r H
\end{array}\right.
$$

Then $\tilde{u}$ solves

$$
\left\{\begin{array}{l}
\tilde{u}_{t t}=\tilde{u}_{r r} \\
\tilde{u}(r, 0)=\tilde{G} \\
\tilde{u}_{t}(r, 0)=\tilde{H}
\end{array}\right.
$$

In other words, even though $\phi$ doesn't solve a wave equation, $\tilde{u}=r \phi$ does!

## Proof:

$$
\begin{aligned}
\tilde{u}_{t t} & =(r \phi)_{t t} \\
& =r \phi_{t t} \\
& =r\left(\phi_{r r}+\left(\frac{n-1}{r}\right) \phi_{r}\right) \quad \text { (By Euler-Poisson-Darboux) } \\
& =r \phi_{r r}+(n-1) \phi_{r} \\
& =r \phi_{r r}+2 \phi_{r} \quad(\text { In 3 dimensions, } n=3) \\
& =\left(r \phi_{r}+\phi\right)_{r} \quad \text { (Product rule) } \\
& =(r \phi)_{r r} \quad \text { (Product rule again) } \\
& =\tilde{u}_{r r}
\end{aligned}
$$

Therefore $\tilde{u}_{t t}=\tilde{u}_{r r}$, and the initial conditions follow from definition.

Upshot: Since $\tilde{u}$ solves a $1 D$ wave equation, we can now use D'Alembert's formula to solve this!

## D'Alembert's Formula

$$
u(x, t)=\frac{1}{2}(g(x-t)+g(x+t))+\frac{1}{2} \int_{x-t}^{x+t} h(s) d s
$$

Here we get:

$$
\begin{aligned}
\tilde{u}(r, t) & =\frac{1}{2}(\tilde{G}(r+t)-\tilde{G}(r-t))+\frac{1}{2} \int_{t-r}^{t+r} \tilde{H}(s) d s \\
r \phi(r, t) & =\frac{1}{2}(\tilde{G}(r+t)-\tilde{G}(r-t))+\frac{1}{2} \int_{t-r}^{t+r} \tilde{H}(s) d s \\
\phi(r, t) & =\frac{1}{2}\left(\frac{\tilde{G}(r+t)}{r}-\frac{\tilde{G}(r-t)}{r}\right)+\frac{1}{2}\left(\frac{\int_{t-r}^{t+r} \tilde{H}(s) d s}{r}\right)
\end{aligned}
$$

Now, letting $r \rightarrow 0$ and using that $\phi(r, t)=f_{\partial B(x, r)} u(y, t) d S(y)$ (the average of $u$ on the sphere centered at $x$ and radius $r$ ), as well as l'Hôpital's rule, we get:

$$
\begin{aligned}
u(x, t) & =\frac{1}{2}\left(\tilde{G}^{\prime}(t)+\tilde{G}^{\prime}(t)\right)+\frac{1}{2}(\tilde{H}(t)+\tilde{H}(t)) \\
& =\tilde{G}^{\prime}(t)+\tilde{H}(t) \\
& =\frac{d}{d t}\left(t f_{\partial B(x, t)} g(y) d S(y)\right)+t f_{\partial B(x, t)} h(y) d S(y) \\
& =f_{\partial B(x, t) g(y) d S(y)}+t \frac{d}{d t}\left(f_{\partial B(x, t)} g(y) d S(y)\right)+t f_{\partial B(x, t)} h(y) d S(y)
\end{aligned}
$$

And using the same trick as usual with the change of variables $z=\frac{y-x}{t}$, the integral with $\frac{d}{d t}$ eventually becomes $f_{\partial B(x, t)} D g(y) \cdot(y-x) d S(y)$.

This finally gives us the following solution for the wave equation in 3 dimensions:

## Kirchoff's Formula $(n=3)$ :

A solution to the wave equation in 3 dimensions is given by

$$
u(x, t)=f_{\partial B(x, t)} t h(y)+g(y)+D g(y) \cdot(y-x) d S(y)
$$

Note: It turns out that a similar (but more complicated) formula holds in odd dimensions (like 5 or 7 dimensions)

## 4. Poisson's Formula and beyond

Reading: Solution for $n=2$ (pages 73-74, optional)
Reading: Section 2.4.2: Nonhomogeneous problem

To find a solution in 2 dimensions, use the following clever trick: Suppose $u\left(x_{1}, x_{2}, t\right)$ solves the wave equation in 2 dimensions, then $\bar{u}\left(x_{1}, x_{2}, x_{3}, t\right)=: u\left(x_{1}, x_{2}, t\right)$ solves the wave equation in 3 dimensions, which means that you can apply Kirchoff's formula to $\bar{u}$ to get
$u\left(x_{1}, x_{2}, t\right)=\bar{u}\left(x_{1}, x_{2}, x_{3}, t\right)=f_{\partial \overline{B(x, t)}} t \bar{h}(y)+\bar{g}(y)+D \bar{g}(y) \cdot(y-x) d S(y)$
(Here $\overline{B(x, t)}$ is the ball in 3 dimensions). But the annoying thing is to simplify everything to get everything back to 2 dimensions.

However, this can be done, see the book if you're interested. The main result is that

## Poisson's Formula $(n=2)$ :

A solution to the wave equation in 2 dimensions is given by

$$
u(x, t)=\frac{1}{2} f_{B(x, t)} \frac{t g(y)+t^{2} h(y)+t D g(y) \cdot(y-x)}{{\sqrt{t^{2}-|y-x|}}^{2}} d y
$$

Note: This trick of using 3 dimensions to solve the 2 dimensional problem is called the method of descent. Similarly, the method of descent can be used to solve the wave equation in even dimensions $(4,6, \ldots)$, which means that we have now solved the wave equation in all dimensions! Wooohoo!!!

Note: Finally, using (what's called) Duhamel's principle (see section 2.4.2 of the book if interested), one can solve the inhomogeneous problem $u_{t t}-\Delta u=f$.

So, for our purposes, we are done with the wave equation, congratulations! And the next lecture will be a sweet surprise $)_{-}$

