LECTURE 3 - FIRST-ORDER LINEAR EQUATIONS (II)

Monday, September 30, 2019 5:23 PM

I- THE GENERAL CASE

The amazing thing is that you can use the same ideas (from last time) to really solve any (linear) first-order PDE!

Example: Solve $\boxed{u_x + y\, u_y = 0}$

STEP 1

$$\Leftrightarrow \quad 1\, u_x + y\, u_y = 0$$

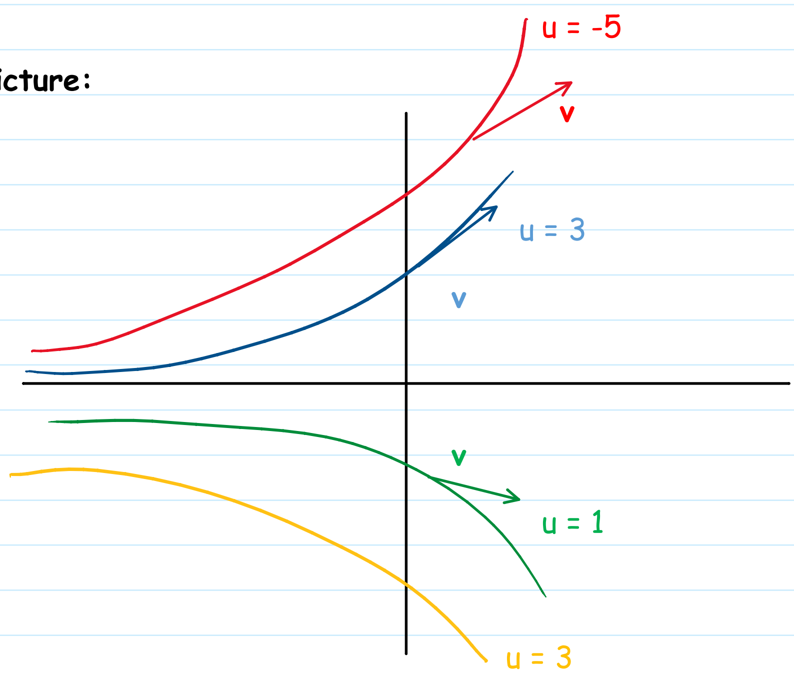
$$\Leftrightarrow \quad (u_x, u_y) \cdot (1, y) = 0$$

$$\Leftrightarrow \quad \nabla u \cdot v = 0 \qquad v = (1, y)$$

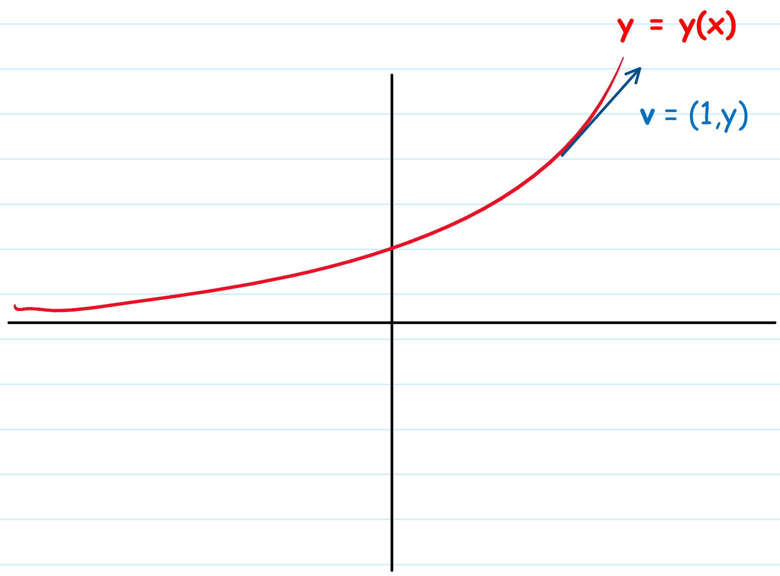
So the directional derivative of u along **v** is still 0, but this time **v** is not constant.

The only difference from the last lecture is that, instead of u being constant on lines, u is constant on curves (with direction/slope **v**)

Picture:



STEP 2: Let's examine a typical curve



On the one hand, since each curve is directed by **v** = (1,y), the slope of each curve should be:

$$y/1 \ \text{(Rise/Run)} = y$$

On the other hand, assuming y is a function of x, y = y(x), its slope is

$$dy/dx \ \text{(Math 2A)}$$

Therefore, we get that: $dy/dx = y$ (ODE)

Solving this using Math 3D, we get: $y = C\, e^x$

Hence: u is constant on curves of the form $y = Ce^x$ (called "characteristic curves")

(Note: So far we have done wishful thinking, but you can indeed check that u is constant on those curves, see the formula on the middle of page 8 in the book)

Therefore:

$$u(x,y) = f(?)$$

Where ? is a variable that is constant on curves of the form $y = Ce^x$

STEP 3

But $y = Ce^x$ implies $y\,e^{-x} = C$

So ? = $y\,e^{-x}$ (which is indeed constant on the curves)

STEP 4: The general solution of our PDE is:

$\boxed{u(x,y) = f(?) = f(ye^{-x})}$ (with f arbitrary)

Example: Solve $\boxed{x\,u_x + y\,u_y = 0}$

1) $\Leftrightarrow \quad (u_x, u_y) \cdot (x,y) = 0$

$\Leftrightarrow \quad \nabla u \cdot v = 0 \qquad v = (x,y)$ (Slope = y/x)

2) This time: $\dfrac{dy}{dx} = \dfrac{y}{x} \quad \left(\Leftrightarrow \quad x\,dy = y\,dx \Leftrightarrow \dfrac{dy}{y} = \dfrac{dx}{x} \right)$

$\Leftrightarrow \int \dfrac{dy}{y} = \int \dfrac{dx}{x}$

$\Leftrightarrow \ln|y| = \ln|x| + C$

$e \qquad\qquad e$

$\Leftrightarrow |y| = |x|\,e^{C}$

$\Leftrightarrow y = \pm x e^{C} = x\,\underbrace{(\pm e^{C})}_{\text{CONSTANT}}$

$\Leftrightarrow y = Cx$

$\Leftrightarrow \dfrac{y}{x} = C = ?$

3) so $\boxed{u(x,y) = f(?) = f\left(\frac{y}{x}\right)}$

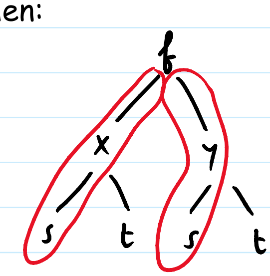
II- REVIEW OF THE CHAIN RULE (CHEN LU)

If you didn't like this method, luckily there is another method (more straightforward, but less intuitive). For this, I need to provide you with the single most important weapon in this course: The Chain Rule

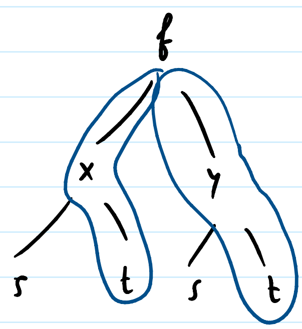
RECALL (MATH 2D):

If f = f(x,y), where x = x(s,t) and y = y(s,t), then:

$$\boxed{\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}}$$



$$\boxed{\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}}$$



(You literally differentiate the hell out of everything: Look at all the slots where s appears, and differentiate f with respect to those slots)

(Make sure to be super comfortable with it, we'll use this many times!)

III- THE COORDINATE METHOD

Different way of solving the equation from the beginning: Less intuitive, but more straightforward.

Ex: Solve $2\, u_x + 3\, u_y = 0$

Trick:

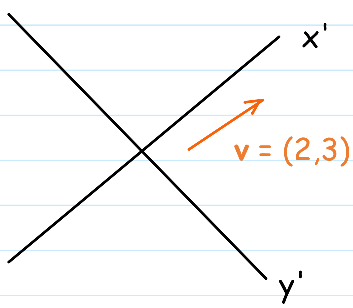
STEP 1: Define new variables x' and y' by:

$$\boxed{\begin{cases} x' = 2x + 3y \\ y' = -3x + 2y \end{cases}}$$

Motivation for this:

$y' = 0$ implies $y = 3/2\, x$ which is the line from before
$x' = 0$ implies $y = -2/3\, x$, which is the line perpendicular to it

(So x' and y' are kind of like new axes)



STEP : Rewrite the PDE $2\, u_x + 3\, u_y = 0$ in terms of x' and y' using the Chen Lu:

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial x'} \frac{\partial x'}{\partial x} + \frac{\partial u}{\partial y'} \frac{\partial y'}{\partial x}$$

$x' = 2x + 3y$
$y' = -3x + 2y$

$$= u_{x'}(2) + u_{y'}(-3)$$

$$u_x = 2 u_{x'} - 3 u_{y'}$$

$$u_y = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial x'} \frac{\partial x'}{\partial y} + \frac{\partial u}{\partial y'} \frac{\partial y'}{\partial y}$$

$$= u_{x'}(3) + u_{y'}(2)$$

$$u_y = 3u_{x'} + 2u_{y'}$$

Then our PDE: $2u_x + 3u_y = 0$

$$\Leftrightarrow 2(2u_{x'} - 3u_{y'}) + 3(3u_{x'} + 2u_{y'}) = 0$$

$$\Leftrightarrow 4u_{x'} - \cancel{6u_{y'}} + 9u_{x'} + \cancel{6u_{y'}} = 0$$

(MOST IMPORTANT STEP)

$$\Leftrightarrow 13u_{x'} = 0$$

$$\Leftrightarrow \boxed{u_{x'} = 0}$$

NEAT!

So in the new variables x' and y', our PDE becomes super easy!!! Hence:

$u_{x'} = 0 \Rightarrow u = f(y')$ (u is constant with respect to x', so only depends on y') (f = arbitrary function)

$$\Rightarrow u(x,y) = f(\underbrace{-3x + 2y}_{y'})$$

$$\Rightarrow \boxed{u(x,y) = f(2y - 3x)}$$ (f arbitrary)

(So the coordinate method is indeed neater, but it requires us to know what the new variables are.)

Moral: If you choose the "right" coordinates, your PDE becomes simplified

(We'll see many instances of this later in the course)

Note: I won't ask you to "guess" what the change of coordinates is (unless in situations where it's not hard to guess), but just know that this method exists and how to use it if I give you the variables.