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Thm) Let $D = \{(t, y) \mid a \le t \le b,\ -\infty < y < \infty\}$, $f(t, y)$ be a continuous function on D.
If f satisfies a Lipschitz condition on D in the variable y, then the IVP

$$\begin{cases} y'(t) = f(t, y), & a \le t \le b, \\ y(a) = \alpha \end{cases}$$

has a unique solution.

$|f(t, y_1) - f(t, y_2)| \le L |y_1 - y_2|$

ex) $y' = y\cos t$, $0 \le t \le 1$, $y(0) = 1$

$$\begin{aligned} |f(t, y_1) - f(t, y_2)| &= |y_1 \cos t - y_2 \sin t| \\ &= |\cos t|\,|y_1 - y_2| \\ &\le |y_1 - y_2| \end{aligned}$$

$\Rightarrow$ Lipschitz on D $(L = 1)$

$\overset{\text{Thm}}{\Rightarrow}$ unique solution

$\hookrightarrow \dfrac{y'}{y} = \cos t \Rightarrow \ln y = \sin t + C_0$

$\Rightarrow y(t) = C_1 e^{\sin t}$

$\Rightarrow y(t) = e^{\sin t}$

$(y(0) = 1)$

ex) $\underline{y' = \frac{2}{t} y + t^2 e^t}$, $1 \le t \le 2$, $y(1) = 0$

nonhomogeneous ODE

$$\begin{aligned} |f(t, y_1) - f(t, y_2)| &= \left|\frac{2}{t}\right| |y_1 - y_2| \\ &\le 2 |y_1 - y_2| \end{aligned}$$

$\Rightarrow$ Lipschitz $(L = 2)$

$\Rightarrow$ unique solution.

$$1 \le t \le 2 \;\Downarrow\; \frac{1}{2} \le \frac{1}{t} \le 1 \;\Downarrow\; 1 \le \frac{2}{t} \le 2$$

$$y' = \frac{2}{t} y + t^2 e^t, \quad y(1) = 0$$

$$\hookrightarrow \quad u y' = \frac{2}{t} u y + u(t^2 e^t)$$

$$u(t) = ?$$

$$(uy)' = uy' + u'y = uy' - \frac{2}{t} u y$$

$$\Rightarrow \quad u' = -\frac{2}{t} u \quad (\text{ODE})$$

$$\Rightarrow \quad \ln u = -2 \ln t$$

$$\Rightarrow \quad u(t) = t^{-2}$$

$$\Rightarrow (uy)' = u(t^2 e^t) = t^{-2}(t^2 e^t) = e^t$$

$$\Rightarrow \quad uy = e^t + C$$

$$\Rightarrow \quad t^{-2} y = e^t - 1 \qquad (y(1) = 0)$$

$$\Rightarrow \quad y(t) = t^2 (e^t - 1)$$

Def) Well-posed problem.

i. unique solution

ii. stability

RHS, initial conditions.

$\hookrightarrow$ small perturbation in given data implies small error in solutions.

A mathematical statement :

small

$$|y(t) - z(t)| \le |f(t,y) - f(t,z)| + |y(a) - z(a)|$$

small

small. gives stability.

Thm) f : continuous, Lipschitz.

Then, the IVP is well-posed.

continuous

ex) $y' = \cos(yt)$, $0 \le t \le 1$, $y(0) = 1$

Well-posedness (Lipschitz)

$$|f(t, y_1) - f(t, y_2)| = |\cos(y_1 t) - \cos(y_2 t)|$$
$$\le L|y_1 - y_2|$$

Another way to prove Lip.

$D = \{(t, y) \mid 0 \le t \le 1, -\infty < y < \infty\}$: convex

$f(t, y)$ is defined on convex D

$\cos(yt)$

$$\Rightarrow \left|\frac{\partial f}{\partial y}(t, y)\right| = |t\sin(yt)| \le 1$$

$\overset{\text{by Thm 5.3}}{\Rightarrow}$ f : Lipschitz in y

Thus, by the theorem, the IVP is well-posed.

exercise) $f(t, y) = e^{t-y} \Rightarrow$ 'Lipschitz ?'