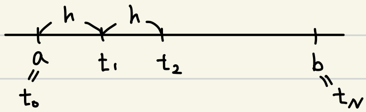
04/07/20

Sec 5.2. Euler's Method

IVP : $\frac{dy}{dt} = f(t, y)$, $a \le t \le b$, $y(a) = \alpha$

Mesh points :



$t_i = a + ih$, $i = 0, 1, \cdots, N$

($h = t_{i+1} - t_i$: step size)

Taylor's theorem gives

$$y(t_{i+1}) = y(t_i) + (t_{i+1} - t_i)\, y'(t_i) + \frac{(t_{i+1} - t_i)^2}{2} y''(\xi_i),$$

for some $\xi_i \in (t_i, t_{i+1})$.

$\Rightarrow$ $y(t_i) \approx w_i$, an approximation with Euler's method.

- (Forward) Euler's method.
  (explicit)

$$w_{i+1} = w_i + h\, y'(t_i) = w_i + h f(t_i, w_i)$$

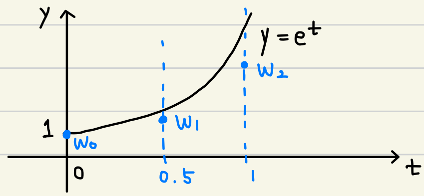
$$w_0 = y(t_0) = \alpha$$

$\left(\begin{array}{c} y'(t) = f(t, y) \\ \downarrow \\ y'(t_i) = f(t_i, y) \end{array} \right)$

ex) $y' = f(t, y) = y$, $0 \le t \le 1$, $y(0) = 1$, $h = 0.5$.

$$w_0 = y(0) = 1 \quad (t=0)$$
$$w_1 = w_0 + hf(t_0, w_0) \quad (t=0.5)$$
$$= 1 + (0.5)\cdot 1 = 1.5 \approx y(0.5)$$
$$w_2 = w_1 + hf(t_1, w_1) \quad (t=1) \quad \text{(one-step method)}$$
$$= 1.5 + (0.5)\cdot(1.5) = 2.25 \approx y(1)$$

$$w_{t=0.25} = \frac{w_0 + w_1}{2}$$



$$y(t) = e^t$$
$$y(0.5) = e^{0.5} \approx 1.64$$
$$y(1) = e^1 \approx 2.7$$

ex) $y' = -y + ty^{1/2}$, $2 \le t \le 3$, $y(2) = 2$, $h = 0.25$

$$w_0 = 2 \quad (t=2)$$
$$w_1 = w_0 + hf(t_0, w_0) \quad (t=2.25)$$
$$= 2 + (0.25)(-2 + 2\sqrt{2}) = 2.207,$$

w_2 $(t=2.5)$, w_3 $(t=2.75)$, w_4 $(t=3)$.

- Error bounds

$$|y(t_{i+1}) - w_{i+1}| \le (1 + hL)\,|y(t_i) - w_i| + \frac{h^2}{2}M$$

'h is small $\Rightarrow$ error is small'

- Backward Euler's method
(implicit)

$$\begin{cases} w_{i+1} = w_i + hf(t_{i+1}, w_{i+1}) \\ w_0 = y(t_0) = \alpha \end{cases}$$

ex) $y' = \underbrace{-100y}_{=f(t,y)}$, $0 \le t \le 2$, $y(0) = 1$. $h = 1$.

(Forward)

$\underset{(t=0)}{w_0} = 1$

$\underset{(t=1)}{w_1} = 1 + 1 f(0, 1) = 1 - 100 = -99$

$\underset{(t=2)}{w_2} = -99 + 1 f(1, -99) = -99 + 9900 = 9801$

(unstable)

$y(t) = e^{-100t} \Rightarrow y(1) = e^{-100} \approx 0$ $= 3.72 \times 10^{-44}$

$y(2) = e^{-200} \approx 0$

0

(Backward) ← reduce comp. time ← solve an equation at each step (expensive step)

$w_0 = 1$

$w_1 = w_0 + h f(t_1, w_1)$

$= 1 + 1 \cdot f(1, w_1) = 1 - 100 w_1$

$\hookrightarrow 101 w_1 = 1 \Rightarrow w_1 = 1/101$

$w_2 = \frac{1}{101} + 1 \cdot f(2, w_2)$

$= \frac{1}{101} - 100 w_2 \Rightarrow w_2 = \frac{1}{(101)^2}$

$w_0 > w_1 > w_2 > 0$ (stable)

'Stability depending on h'

exercise) $y' = -100y$, $0 \le t \le 2$, $y(0) = 1$, $h = 0.001$