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Sec 5.3. High-Order Taylor Methods

$$y' = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha.$$

Def) Local truncation error

$$\begin{cases} w_0 = \alpha \\ w_{i+1} = w_i + h\phi(t_i, w_i) \end{cases}$$

$$\Rightarrow T_{i+1}(h) = \frac{y_{i+1} - (y_i + h\phi(t_i, y_i))}{h}$$

$y_{i+1} \approx y_i + h\phi(t_i, y_i)$ from Taylor.

ex) Backward Euler

$$\begin{cases} w_0 = \alpha \\ w_{i+1} = w_i + hf(t_{i+1}, w_{i+1}) \end{cases}$$

$y_{i+1} \approx y_i + hf(t_{i+1}, w_{i+1})$

$h^2 = (t_i - t_{i+1})^2$

$$\Rightarrow T_{i+1}(h) = \frac{y_{i+1} - (y_i + hf(t_{i+1}, w_{i+1}))}{h}$$

$y(t_i) = y(t_{i+1}) + (t_i - t_{i+1})y'(t_{i+1}) + \frac{h^2}{2}y''(\xi_i)$, $\quad (t_i - t_{i+1}) = -h$

$$= \frac{y(t_{i+1}) - y(t_i) - hy'(t_{i+1})}{h}$$

$$= -\frac{h^2}{2}y''(\xi_i) \Big/ h, \quad \xi_i \in (t_i, t_{i+1})$$

$$= -\frac{h}{2}y''(\xi_i)$$

- Taylor methods (explicit)

$$y(t_{i+1}) = y(t_i) + hy'(t_i) + \cdots + \frac{h^n}{n!}y^{(n)}(t_i) + \frac{h^{n+1}}{(n+1)!}y^{(n+1)}(\xi_i)$$

'Order 2'

$$\begin{cases} w_0 = \alpha \\ w_{i+1} = w_i + h f(t_i, w_i) + \frac{h^2}{2} f'(t_i, w_i) \end{cases}$$

$$\Rightarrow T_{i+1}(h) = \frac{h^2}{3!} y^{(3)}(\xi_i) = O(h^2)$$

ex) $y' = \underbrace{te^{3t} - 2y}_{= f(t,y)}$, $0 \le t \le 1$, $y(0) = 0$, $h = 0.5$

$$\hookrightarrow f'(t, y) = e^{3t} + 3te^{3t} - 2y'$$
$$= e^{3t} + 3te^{3t} - 2f(t, y)$$

$$\underset{(t=0)}{w_0} = 0$$

$$\underset{(t=0.5)}{w_1} = 0 + h f(0, 0) + \frac{h^2}{2} f'(0, 0)$$
$$= 0 + (0.5) \cdot 0 + \frac{(0.5)^2}{2} \cdot 1$$
$$= 0.125$$

$$w_2 = 0.125 + (0.5) f(0.5, 0.125) + \frac{(0.5)^2}{2} f'(0.5, 0.125)$$

- Explicit methods $\Rightarrow$ need small h.
  Implicit methods $\Rightarrow$ not need small h, but need to solve an equation at each step.

- Backward ? Taylor Methods ?

$$\begin{cases} w_0 = \alpha \\ w_{i+1} = w_i + \underline{h f(t_{i+1}, w_{i+1}) - \frac{h^2}{2} f'(t_{i+1}, w_{i+1})} \end{cases}$$

The equation at each iteration would be very complicated due to f'.

- Crank - Nicolson (implicit)

$$\begin{cases} w_0 = \alpha \\ w_{i+1} = w_i + \frac{h}{2}\left(f(t_i, w_i) + f(t_{i+1}, w_{i+1})\right) \end{cases}$$

$$\Rightarrow \quad T_{i+1}(h) = o(h^2)$$

Quiz) $y' = \frac{1+y}{1+t}$, $1 \le t \le 2$, $y(1) = 2$

1. Show that the IVP is well-posed.

2. Use Euler's method to approximate the solution for the IVP with $h = 0.5$.