04/14/20

Sec 5.4. Runge - Kutta Methods (explicit)

- Main idea (RK 2)

$f' = \frac{df}{dt}$

$$T^{(2)}(t,y) = f(t,y) + \frac{h}{2} \underline{f'(t,y)}$$

↑ not efficient

$$= f(t,y) + \frac{h}{2} \cdot \frac{\partial f}{\partial t}(t,y) + \frac{h}{2} \cdot \frac{\partial f}{\partial y}(t,y) \cdot f(t,y)$$

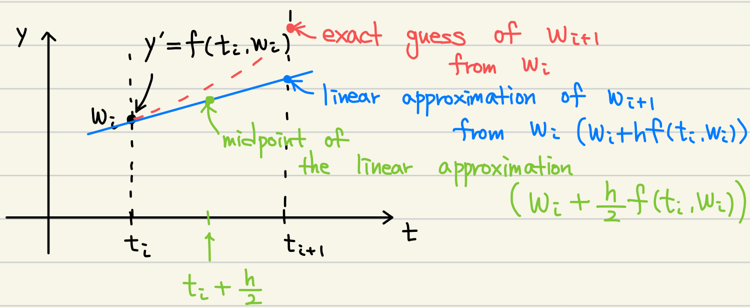
error $O(h^2)$

$$a_1 f(t+\alpha_1, y+\beta_1) = a_1 f(t,y) + a_1\alpha_1 \frac{\partial f}{\partial t}(t,y) + a_1\beta_1 \frac{\partial f}{\partial y}(t,y)$$

$$\Rightarrow a_1 = 1, \quad \alpha_1 = \frac{h}{2}, \quad \beta_1 = \frac{h}{2} f(t,y)$$

- RK 2 method (midpoint method)

$$\begin{cases} w_0 = \alpha \\ w_{i+1} = w_i + hf\left(t_i + \frac{h}{2}, w_i + \frac{h}{2} f(t_i, w_i)\right) \end{cases}$$



- Modified Euler method (explicit)

$$a_1 f(t, y) + a_2 f(t + \alpha_2, y + \delta_2)$$

$$w_0 = \alpha$$

$$w_{i+1} = w_i + \frac{h}{2}\left[f(t_i, w_i) + f(t_{i+1}, w_{i+1}) \right]$$

$w_{i+1} = w_i + h f(t_i, w_i)$

(Euler method
linear approximation)

ex) $y = 1 + (t-y)^2$, $2 \le t \le 3$, $y(2) = 1$, $h = 0.5$.

- RK 2 $(O(h^2))$

$$w_0 = 1$$

$$w_1 = w_0 + h f\left(t_0 + \frac{h}{2},\ w_0 + \frac{h}{2} f(t_0, w_0) \right)$$

$f(2,1) = 2$

$$= 1 + \frac{1}{2} f\left(\frac{9}{4}, \frac{3}{2} \right)$$

- Modified Euler $(O(h^2))$

$$w_0 = 1$$

$$w_1 = w_0 + \frac{h}{2}\left[f(t_0, w_0) + f(t_1, w_0 + h f(t_0, w_0)) \right]$$

$f(t_0, w_0) = 2$, $t_1 = t_0 + h$, $f(t_0, w_0) = 2$

$$= 1 + \frac{1}{4}\left[2 + f(2.5, 2) \right]$$

- Different expression for RK methods

- RK 2

$$\begin{cases} w_0 = \alpha \\ w_{i+1} = w_i + hf\left(t_i + \frac{h}{2},\ w_i + \frac{h}{2} f(t_i, w_i)\right) \end{cases}$$

$$\Longleftrightarrow \quad \begin{aligned} w_0 &= \alpha \\ k_1 &= hf(t_i, w_i) \\ w_{i+1} &= w_i + hf\left(t_i + \frac{h}{2},\ w_i + \frac{k_1}{2}\right) \end{aligned}$$

- RK 3 (Heun's method, $O(h^3)$)

$$w_0 = \alpha$$

$$k_1 = hf(t_i, w_i)$$

$$k_2 = hf\left(t_i + \frac{h}{3},\ w_i + \frac{1}{3} k_1\right)$$

$$k_3 = hf\left(t_i + \frac{2h}{3},\ w_i + \frac{2}{3} k_2\right)$$

$$w_{i+1} = w_i + \frac{1}{4}(k_1 + 3k_3)$$