04/21/20

Sec 5.6. Multistep Methods

- A new approach (Numerical integration)

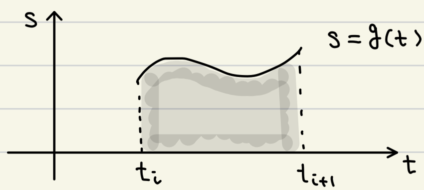
$y' = f(t, y)$

$$\Rightarrow \int_{t_i}^{t_{i+1}} y'(t)\,dt = \int_{t_i}^{t_{i+1}} f(t, y)\,dt$$

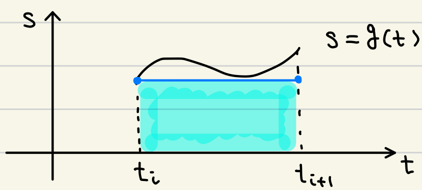
$$\overset{\text{FTC}}{\Rightarrow} y(t_{i+1}) - y(t_i) = \int_{t_i}^{t_{i+1}} \overbrace{f(t, y)}^{=: g(t)}\,dt$$

$$i=0 \quad \Big(y(t_1) - \underbrace{y(t_0)}_{\text{initial}} = \int_{t_0}^{t_1} g(t)\,dt \Big)$$

How can we (approximate) compute $\int_{t_i}^{t_{i+1}} g(t)\,dt$?



ex) Picking the left end point



$$\int_{t_i}^{t_{i+1}} g(t)\,dt \approx (t_{i+1} - t_i)\, g(t_i)$$

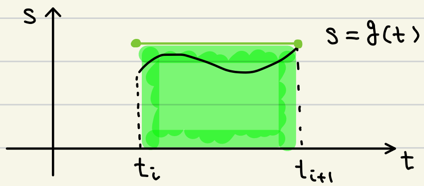
$$= h f(t_i, y(t_i))$$

$$\Rightarrow \quad w_1 = w_0 + hf(t_0, y(t_0)) = w_0 + hf(t_0, w_0)$$

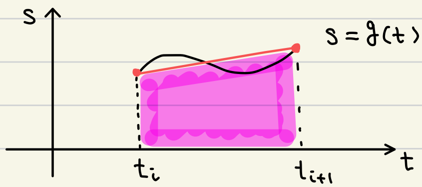
(Forward Euler)

'There are cumulative errors'
$(y(t_1) \approx w_1)$

exercise) The right end point.



ex) Traperzoidal rule



$$\int_{t_i}^{t_{i+1}} g(t)\,dt \approx \frac{h}{2}\left(g(t_i) + g(t_{i+1})\right) = \frac{h}{2}\left(f(t_i, y(t_i)) + f(t_{i+1}, y(t_{i+1}))\right)$$

(Crank - Nicolson)

Moreover,

$$\int_{t_i}^{t_{i+1}} g(t)\,dt \approx \int_{t_i}^{t_{i+1}} P_1(t)\,dt$$

a first-order polynomial for
the line segment $(at+b)$

$$P_1(t) = at + b$$
$$= \frac{g(t_{i+1}) - g(t_i)}{t_{i+1} - t_i}\, t + \frac{g(t_i)t_{i+1} - g(t_{i+1})t_i}{t_{i+1} - t_i}$$

(Why? $P_1(t_i) = at_i + b = g(t_i)$
$P_1(t_{i+1}) = at_{i+1} + b = g(t_{i+1})$)
'Interpolation'

- Multistep methods

$$\int_{t_i}^{t_{i+1}} g(t)\,dt \approx \int_{t_i}^{t_{i+1}} P_2(t)\,dt$$

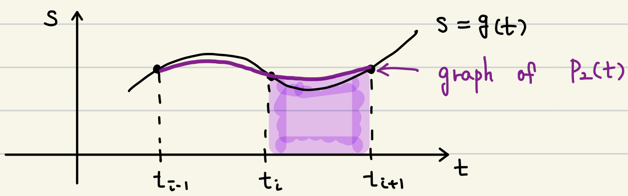
$P_2(t) = at^2 + bt + c$

$\hookrightarrow$ $P_2(t_i) = at_i^2 + bt_i + c = g(t_i)$

$P_2(t_{i+1}) = at_{i+1}^2 + bt_{i+1} + c = g(t_{i+1})$

'One more equation!'

$P_2(t_{i-1}) = at_{i-1}^2 + bt_{i-1} + c = g(t_{i-1})$

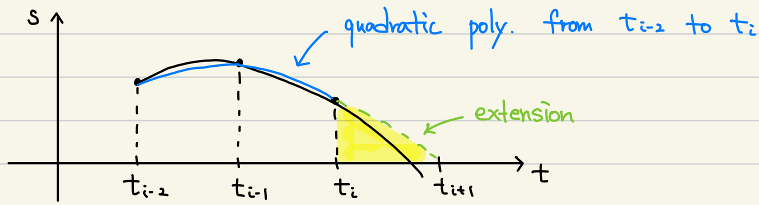


$$\Rightarrow \quad w_{i+1} - w_i = \int_{t_i}^{t_{i+1}} at^2 + bt + c\,dt$$

(Adams - Moulton (two-step))

- $P_2(t_{i+1}) \Rightarrow$ Implicit methods!

- $$P_2(t_{i-2}) = g(t_{i-2})$$
  $$P_2(t_{i-1}) = g(t_{i-1})$$
  $$P_2(t_i) = g(t_i)$$



$$\int_{t_i}^{t_{i+1}} g(t)\,dt \approx A \text{ (yellow)}$$
$$= \int_{t_i}^{t_{i+1}} P_2(t)\,dt$$

(Adams - Bashforth (three-step))

- No $P_2(t_{i+1})$ $\Rightarrow$ explicit methods !