04/28/20

ex) $u_1' = u_2$, $u_1(0) = 1$

$u_2' = -u_1 - 2e^t + 1$, $u_2(0) = 0$.

$0 \le t \le 2$, $h = 0.5$

↳ Euler's method.

$$\vec{w}_{i+1} = \vec{w}_i + h\vec{f}(t_i, \vec{w}_i)$$

$$\Rightarrow \quad w_1^1 = w_0^1 + hf_1(t_0, w_0^1, w_0^2)$$
$$= 1 + h w_0^2 = 1 + (0.5)\cdot 0 = 1$$
$$w_1^2 = w_0^2 + hf_2(t_0, w_0^1, w_0^2)$$
$$= 0 + (0.5)(-w_0^1 - 2e^{t_0} + 1)$$
$$= (0.5)(-1 - 2 + 1) = -1$$

ex) $y'' - 3y' + 2y = 6e^{-t}$, $0 \le t \le 1$,

$y(0) = y'(0) = 2$, $h = 0.1$.

$\Rightarrow$ corresponding system

$\boxed{u_1' = u_2}$ ($y = u_1$, $y' = u_2$)

$$u_2' = 3y' - 2y + 6e^{-t}$$
$$= 3u_2 - 2u_1 + 6e^{-t}$$

→ In the previous example,

$y'' = (u_2') = -u_1 - 2e^t + 1$

$\Rightarrow y'' = -y - 2e^t + 1$ $\quad y(0) = 1$, $y'(0) = 0$

Sec 5.10. Stability

Stability (global) + Consistency (local) $\Rightarrow$ Convergence

- One-step methods

  - Consistency ($h = 0 \Rightarrow \phi = f$)

$$\lim_{h\to 0} \max_{1\le i\le N} |\tau_i(h)| = 0$$

  - Stability

RHS : $\phi(t_i, w_i, h)$
initial : w_0

small perturbation in given data implies small changes in numerical solutions.

  - Convergence

$$\lim_{h\to 0} \max_{1\le i\le N} |w_i - y(t_i)| = 0$$

Thm) IVP : $y' = f(t, y)$, $a \le t \le b$, $y(a) = \alpha$

One-step difference method :

$$\begin{cases} w_0 = \alpha \\ w_{i+1} = w_i + h\phi(t_i, w_i, h) \end{cases}$$

with ϕ is continuous and Lipschitz (w)
on $D = \{ a \le t \le b,\ -\infty < w < \infty,\ 0 \le h \le h_0 \}$.
Then, the method is stable, and

$$|y(t_i) - w_i| \le C\tau(h),$$

where $|\tau_i(h)| \le \tau(h)$.

$$\hookrightarrow \quad u_i - v_i = u_{i-1} + h\phi(t_{i-1}, u_{i-1}, h) - v_{i-1} - h\phi(t_{i-1}, v_{i-1}, h)$$

Lipschitz

$$\leq |u_{i-1} - v_{i-1}| + hL|u_{i-1} - v_{i-1}|$$

$$= (1+hL)|u_{i-1} - v_{i-1}|$$

$$\Rightarrow |u_i - v_i| \leq (1+hL)^i |u_0 - v_0|$$

perturbed problem

- Multistep methods

• Consistency

$$\lim_{h\to 0} |\tau_i(h)| = 0, \quad i = m, m+1, \cdots, N$$

$$\lim_{h\to 0} |\alpha_i - y(t_i)| = 0, \quad i = 1, 2, \cdots, m-1$$

• Stability (root condition)

$$w_{i+1} = a_{m-1}w_i + a_{m-2}w_{i-1} + \cdots + a_0 w_{i+1-m} + hF(t_i, h, \text{—})$$

$$\Rightarrow \quad P(\lambda) = \lambda^m - a_{m-1}\lambda^{m-1} - a_{m-2}\lambda^{m-2} - \cdots - a_0$$

Let $\lambda_1, \cdots, \lambda_m$ be its roots.

i) $|\lambda_i| \leq 1$

ii) if $|\lambda_i| = 1$, λ_i is a simple root (multiplicity : 1)

'Root condition'

ex) A-B two-step explicit method

$$w_{i+1} = w_i + hF(\text{—})$$

$\Rightarrow$ $P(\lambda) = \lambda - 1$

$\Rightarrow$ $\lambda = 1$

$\Rightarrow$ (strongly) stable.

ex) Simpson's method

$$w_{i+1} = w_{i-1} + hF(\text{—})$$

$\Rightarrow$ $P(\lambda) = \lambda^2 - 1 = (\lambda - 1)(\lambda + 1)$

$\Rightarrow$ $\lambda = 1, -1$

$\Rightarrow$ (weakly) stable

ex) $w_{i+2} = 4w_{i+1} - 3w_i - 2hf(t_i, w_i)$

$\Rightarrow$ $P(\lambda) = \lambda^2 - 4\lambda + 3$

$\qquad = (\lambda - 3)(\lambda - 1)$

$\Rightarrow$ $\lambda = 1, 3$

$\Rightarrow$ unstable