04/30/20

Sec 5.11. Stiff Differential Equations

- Main issue

  - IVP : $y' = -100y \ , \quad y(0) = 1$

    $\Rightarrow \ y(t) = e^{-100t}$

    ↳ well-posed problem

  - Forward Euler

    $$\begin{cases} w_0 = \alpha \\ w_{i+1} = w_i + hf(t_i, w_i) \end{cases}$$

    ↳ stable, consistent $(h \to 0)$

    $$y(t_1) \approx w_1 = w_0 + hf(t_0, w_0)$$

    $$\Rightarrow |y(t_1) - w_1| \le \frac{h^2}{2} |y''(\xi_1)| ,$$

    where $t_0 < \xi_1 < t_1$.

  - $y''(t) = 100^2 e^{-100t}$.

    If $h = 0.1$ and $\xi_1 = 0.01$, then

    $$|y(t_1) - w_1| \le \frac{(0.1)^2}{2} \cdot \frac{(100)^2}{e} \approx 18$$

    ↳ can't guarantee a good approximation w_1 with $h = 0.1$.

    ↳ with sufficiently small h, we can get the convergence.

"How can we get a proper step size h depending on λ such that $y' = \lambda y$?"

↳ $\lambda \ll 0 \Rightarrow$ a stiff equation.

- One-step methods

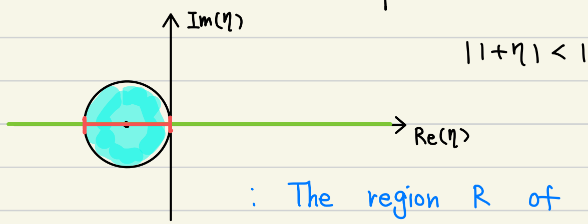
$y' = \lambda y$, $\quad y(0) = \alpha$: a test problem

- Forward Euler

$$w_{i+1} = (1 + h\lambda) w_i = (1 + h\lambda)^{i+1} \alpha$$

$$\Rightarrow \quad |1 + h\lambda| < 1$$: stability condition (for given h, good approximation)

Let $\eta = h\lambda$ be a complex number.



: The region R of absolute stability.

If $\lambda < 0$, $\quad Im(\eta) = 0$, so $R = (-2, 0)$.

$$\Rightarrow \quad -2 < \lambda h < 0$$

$$\Rightarrow \quad h < -(2/\lambda)$$

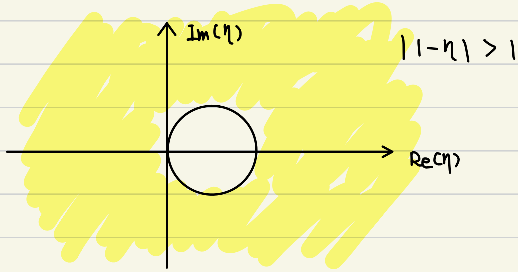
(If $\lambda = -100$, $\quad h < 0.02$)

- Backward Euler

$$w_{i+1} = w_i + h\lambda w_{i+1}$$

$$\Rightarrow \quad \frac{1}{|1 - h\lambda|} < 1$$

$$\Rightarrow \quad |1 - h\lambda| > 1 \quad (|1 - \eta| > 1)$$



If $Re(\eta) < 0$, then it is absolutely stable.
If $\eta < 0$, it is absolutely stable.
: A-stable ($\{\eta \in \mathbb{C} \mid Re(\eta) < 0\} \subset R$)

- Mutistep methods

- characteristic polynomial
  (different from Sec 5.10)
  '$P(z, \eta)$'

- Stability region
  Let $z_1(\eta)$, $z_2(\eta)$ be zeros of $P(z, \eta)$.
  Then,
  $$R = \{\eta \in \mathbb{C} \mid |z_1(\eta)| < 1,\ |z_2(\eta)| < 1\}.$$

ex) A-B explicit method

$$w_{i+1} = w_i + \frac{h}{2}(3\lambda w_i - \lambda w_{i-1})$$

$$= \left(1 + \frac{3h\lambda}{2}\right) w_i - \frac{h\lambda}{2} w_{i-1}$$

$$\Rightarrow \quad P(z,\eta) = z^2 - \left(1 + \frac{3\eta}{2}\right) z + \frac{\eta}{2}$$

$$\Rightarrow \quad z = \left(\frac{1}{2} + \frac{3\eta}{4}\right) \pm \frac{1}{2}\sqrt{1 + \eta + \frac{9\eta^2}{4}}$$

$$\Rightarrow \quad |z|^2 = \left(\frac{1}{2} + \frac{3\eta}{4}\right)^2 + \frac{1}{4}\left(1 + \eta + \frac{9\eta^2}{4}\right)$$

$$= \frac{1}{2} + \eta + \frac{9\eta^2}{4} < 1$$

(In midterm, $\eta < 0$: a real number)

If $P(z,\eta)$ is of order 1, then the method is an one-step method.