05/05/20

Chap. 11. Boundary-Value Problems

- General BVPs.

$$y'' = f(x, y, y'), \quad a \le x \le b$$
$$\text{with} \quad y(a) = \alpha, \quad y(b) = \beta$$

ex) $$y'' = 2yy', \quad 1 \le x \le 2$$
$$\text{with} \quad y(1) = -1, \quad y(2) = -1/2.$$

Analytic solution

$$\Rightarrow \quad y'' = (y^2)'$$
$$\Rightarrow \quad y' = y^2 + C = y^2 \qquad (y(1) = -1)$$
$$\Rightarrow \quad \frac{1}{y^2} \cdot \frac{dy}{dx} = 1$$
$$\Rightarrow \quad -\frac{1}{y} = x + C \qquad (y(2) = -1/2)$$
$$\Rightarrow \quad y(x) = -\frac{1}{x}$$

- Linear BVPs

$$y'' = p(x)y' + q(x)y + r(x), \quad a \le x \le b$$
$$\text{with} \quad y(a) = \alpha, \quad y(b) = \beta.$$

ex> $$y'' = -y, \quad 0 \le x \le \pi/2$$
$$\text{with} \quad y(0) = 0, \quad y(\pi/2) = 1$$

$$\Rightarrow \quad y(x) = A\cos x + B\sin x$$
$$\Rightarrow \quad y(x) = \sin x$$

Sec 11.1. The Linear Shooting Method

$$y'' = p(x)y' + q(x)y + r(x)$$

- Equivalent system of ODEs (IVPs)

$$\begin{cases} y_1'' = p(x)y_1' + q(x)y_1 + r(x), \\ \qquad y_1(a) = \alpha, \quad y_1'(a) = 0 \\ y_2'' = p(x)y_2' + q(x)y_2 \\ \qquad y_2(a) = 0, \quad y_2'(a) = 1 \end{cases}$$

$$y(x) = y_1(x) + \frac{\beta - y_1(b)}{y_2(b)} y_2(x)$$

'BVP $\longrightarrow$ IVPs $\longrightarrow$ Numerical methods'
(Euler, RK4)

ex) $y'' = 4y - 4x$, $0 \le x \le 1$, $y(0) = 0$, $y(1) = 2$,
$h = 1/2$.

equivalent system $\Rightarrow$ $\begin{cases} y_1'' = 4y_1 - 4x, & y_1(0) = 0, & y_1'(0) = 0 \\ y_2'' = 4y_2, & y_2(0) = 0, & y_2'(0) = 1 \end{cases}$

First-order system $\Rightarrow$ $\begin{cases} u_1' = u_2 \\ u_2' = 4u_1 - 4x \end{cases} \cdots (i)$ $\quad \begin{cases} v_1' = v_2 \\ v_2' = 4v_1 \end{cases} \cdots (ii)$

$u_1(0) = 0$, $u_2(0) = 0$ $\qquad v_1(0) = 0$, $v_2(0) = 1$

Euler

$$\Rightarrow \quad u_{1,1} = u_{1,0} + h u_{2,0} = 0 + h \cdot 0 = 0$$

$$u_{2,1} = u_{2,0} + h(4u_{1,0} - 4x_0) = 0 + h \cdot 0 = 0$$

$$\hookrightarrow \quad u_{1,2} = u_{1,1} + h u_{2,1} = 0$$

$$u_{2,2} = u_{2,1} + h(4u_{1,1} - 4x_1) = -1$$

$$v_{1,1} = v_{1,0} + h v_{2,0} = 0 + (0.5) \cdot 1 = 0.5$$

$$v_{2,1} = v_{2,0} + h(4v_{1,0}) = 1$$

$$\hookrightarrow \quad v_{1,2} = v_{1,1} + h v_{2,1} = 1$$

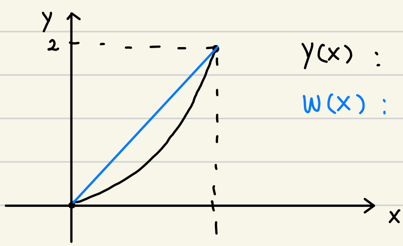
$$v_{2,2} = v_{2,1} + h(4v_{1,1}) = 2$$

$$\Rightarrow \quad y(0.5) = y_1(0.5) + \frac{\beta - y_1(1)}{y_2(1)} y_2(0.5)$$

$$\approx u_{1,1} + \frac{2 - u_{1,2}}{v_{1,2}} v_{1,1}$$

$$= 0 + \frac{2-0}{1}(0.5) = 1$$

$$\Rightarrow \quad y(0) = 0 \quad , \quad y(0.5) \approx 1 \quad , \quad y(1) = 2$$



$y(x)$: exact.

$w(x)$: approx.

'There is error at $x = 0.5$'

Sec 11.2. The Shooting Method for Nonlinear Problems

$$y'' = f(x, y, y'), \quad a \le x \le b$$
$$y(a) = \alpha, \quad \underline{y(b) = \beta}$$
$$\Downarrow$$
$$y'(a) = t_\beta$$

'BVP $\longrightarrow$ IVP'

Let $y(x,t)$ be the solution to the IVP.
$(y(a) = \alpha, \ y'(a) = t)$

Let $w_i(t) \approx y(x_i, t)$ from a numerical method.

Goal: Find t_β such that $w_N(t_\beta) = \beta$

ex) $y'' = y^2, \quad 0 \le x \le 1, \quad y(0) = 0, \quad y(1) = 2.$
$h = 1/2.$

First-order system

$$\Rightarrow \begin{cases} u_1' = u_2 \\ u_2' = u_1^2 \end{cases}$$
$$u_1(0) = 0, \quad u_2(0) = \text{'}t\text{'}$$

Euler

$$\Rightarrow \quad u_{1,1} = u_{1,0} + h u_{2,0} = th$$
$$u_{2,1} = u_{2,0} + h u_{1,0}^2 = t$$
$$\hookrightarrow \quad u_{1,2} = u_{1,1} + h u_{2,1} = 2th$$
$$u_{2,2} = u_{2,1} + h u_{1,1}^2 = h^3 t^2 + t$$

$$\Rightarrow \quad y(1) = 2 \approx u_{1,2} = 2th = t \quad (t_\beta = 2)$$

$$\Rightarrow \quad y(0) = 0, \quad y(0.5) \approx u_{1,1} = 1, \quad y(1) = 2$$