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$$y(b, t_\beta) = \beta \quad \Rightarrow \quad W_{1,N}(t_\beta) = \beta$$

Let $g(t) = W_{1,N}(t) - \beta$. Then, we need to solve $g(t) = 0$ $(t = t_\beta)$

- Bisection method

$$g(t_0) \cdot g(t_1) < 0 \qquad \text{(IVT)}$$

$$\Rightarrow t_2 = (t_0 + t_1)/2 \ : \ \text{midpoint}$$

Compare $g(t_0)g(t_2)$ with $g(t_1)g(t_2)$.

If $g(t_0)g(t_2) < 0$, then $t_3 = (t_0 + t_2)/2$

$$t_0, t_1, t_2, \cdots \rightarrow t_\beta$$

'Each iteration needs a numerical method to get $W_{1,N}(t_k)$ $(g(t_k))$'

- Secant method

$$t_{k+1} = t_k - \frac{g(t_k)(t_k - t_{k-1})}{g(t_k) - g(t_{k-1})}$$

$$= t_k - \frac{(W_{1,N}(t_k) - \beta)(t_k - t_{k-1})}{W_{1,N}(t_k) - W_{1,N}(t_{k-1})}$$

- Newton's method

$$t_{k+1} = t_k - \frac{g(t_k)}{(dg/dt)(t_k)}$$
$$= t_k - \frac{w_{1,N}(t_k) - \beta}{\boxed{(dg/dt)(t_k)}}$$

How can we get this?

$$\frac{d}{dt}(g(t)) = \frac{d}{dt}(w_{1,N}(t))$$

We don't know the function $w_{1,N}(t)$, and we only know $w_{1,N}(t_0), \cdots, w_{1,N}(t_k)$, pointwisely defined.

- $\frac{d}{dt}(y(b,t))$ $\quad (y(b,t) \approx w_{1,N}(t))$

Let $z = dy/dt$. Then,

$z' = \frac{dz}{dx}$

$$z'' = \frac{\partial f}{\partial y} z + \frac{\partial f}{\partial y'} z', \quad a \le x \le b$$

with $z(a) = 0$, $z'(a) = 1$ from the original problem.

ex) $y'' = 2y^3$, $-1 \le x \le 0$, $y(-1) = 1/2$, $y(0) = 1/3$.

Shooting method with Newton's method.

$\Rightarrow$ $z'' = 6y^2 z$, $z(-1) = 0$, $z'(-1) = 1$

First-order system $\Rightarrow$

$$z_1' = z_2$$
$$z_2' = 6y^2 z_1$$

Euler $\Rightarrow$

$$z_{1,1} = z_{1,0} + h z_{2,0} = h \qquad (x = x_0 = -1)$$
$$z_{2,1} = z_{2,0} + h(6 y^2 z_{1,0}) = 1$$

y^2 = $(y(-1))^2 = (1/2)^2$

$$\hookrightarrow \; z_{1,2} = z_{1,1} + h z_{2,1} = 2h$$
$$z_{2,2} = z_{2,1} + h(6y^2 z_{1,1}) \qquad (x = x_1 = -1+h)$$
$$= 1 + 6h^2 y^2$$

y^2 = $(y(-1+h))^2$ ← exact

$\simeq (w_{1,1}(t))^2$

Let z_h be a numerical solution for z using the above procedure. Then, z_h is depending on t because we approximate $y(x_i) \approx w_{1,i}(t)$ and $y'(x_i) \approx w_{2,i}(t)$.

Hence,

$$t_{k+1} = t_k - \frac{w_{1,N}(t_k) - \beta}{z_h(t_k)}$$

" Each iteration needs numerical methods to get $w_{1,N}(t_k)$ and $z_h(t_k)$. "