05/12/20

Sec 11.3. Finite-Difference Methods for Linear Problems.

$$y'' = p(x)y' + q(x)y + r(x),$$
for $a \le x \le b$, $y(a) = \alpha$, $y(b) = \beta$

$$y'' \approx \frac{w_{i+1} - 2w_i + w_{i-1}}{h^2}$$

$$y' \approx \frac{w_{i+1} - w_{i-1}}{2h} \qquad \left(\frac{w_{i+1} - w_i}{h}, \quad \frac{w_i - w_{i-1}}{h} \right)$$

$$y \approx w_i$$

ex) $y'' = 4(y - x)$, $0 \le x \le 1$
$y(0) = 0$, $y(1) = 2$, $h = 0.25$.

$x_0 = 0$ $\quad x_1 \quad x_2 \quad x_3 \quad 1 = x_4$

$$x_1: \quad \frac{w_2 - 2w_1 + \overset{=0}{w_0}}{h^2} = 4w_1 - 4x_1$$

$$x_2: \quad \frac{w_3 - 2w_2 + w_1}{h^2} = 4w_2 - 4x_2$$

$$x_3: \quad \frac{\overset{=2}{w_4} - 2w_3 + w_2}{h^2} = 4w_3 - 4x_3$$

$$\Rightarrow \quad x_1: \quad -(4h^2+2)w_1 + w_2 = \overbrace{-4h^2x_1}^{\text{known}}$$
$$x_2: \quad w_1 - (4h^2+2)w_2 + w_3 = -4h^2x_2$$
$$x_3: \quad w_2 - (4h^2+2)w_3 = -4h^2x_3 - 2$$

$$\Rightarrow \quad A\vec{w} = \vec{b}$$

$$A = \begin{bmatrix} -(4h^2+2) & 1 & 0 \\ 1 & -(4h^2+2) & 1 \\ 0 & 1 & -(4h^2+2) \end{bmatrix}$$

$$\vec{w} = [\, w_1 \quad w_2 \quad w_3 \,]^T$$

$$\vec{b} = [\, -4h^2x_1 \quad -4h^2x_2 \quad -4h^2x_3 - 2 \,]^T$$

$$\Rightarrow \quad \vec{w} = A^{-1}\vec{b} \qquad (A\backslash\vec{b} \text{ in MATLAB})$$

(There are linear - algebra issues)

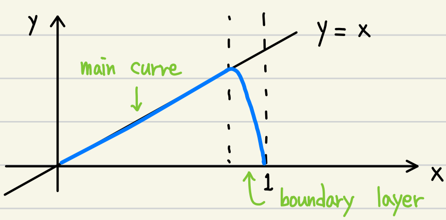
Thm) Suppose that p, q, r are continuous on $[a, b]$. If $q(x) \geq 0$ on $[a, b]$, then the tridiagonal linear system has a unique solution provided that $h < 2/L$, where $L = \max_{a \leq x \leq b} |p(x)|$.

(off-diagonal entries must be nonpositive.)

ex) $-\varepsilon y'' + y' = 1$ in $(0, 1)$ $\quad (L = 1/\varepsilon)$

$y(0) = 0, \quad y(1) = 0$

$$y(x) = x - \frac{e^{-(1-x)/\varepsilon} - e^{-1/\varepsilon}}{1 - e^{-1/\varepsilon}}$$



Why? If $\varepsilon \approx 0$, $y' \approx 1$, so

$y \approx x$ with $y(0) = 0$

However, the boundary condition $y(1) = 0$ must be satisfied.

- FDM $(h = 1/3)$

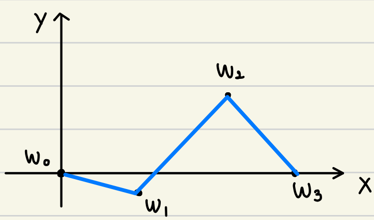
$$x_1: \quad -\varepsilon \frac{w_2 - 2w_1 - w_0}{h^2} + \frac{w_2 - w_0}{h} = 1$$

$$x_2: \quad -\varepsilon \frac{w_3 - 2w_2 - w_1}{h^2} + \frac{w_3 - w_1}{h} = 1$$

$$\Rightarrow \quad x_1: \quad 2\varepsilon w_1 + (h - \varepsilon) w_2 = h^2$$
$$x_2: \quad -(h+\varepsilon) w_1 + 2\varepsilon w_2 = h^2$$

$$\Rightarrow \quad A = \begin{bmatrix} 2\varepsilon & h - \varepsilon \\ -h - \varepsilon & 2\varepsilon \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} h^2 \\ h^2 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = A^{-1}\vec{b} \approx \begin{bmatrix} -0.0262 \\ 0.4987 \end{bmatrix} \quad (\varepsilon = 0.1)$$



Why? $h = 1/3 > 0.2 = 2\varepsilon = 2/L$

↳ If $\varepsilon = 0.1$, then

$h < 2/L = 2\varepsilon = 0.2$

– Upwind method

$$x_1: \quad -\varepsilon \frac{w_2 - 2w_1 - w_0}{h^2} + \frac{w_1 - \overset{\text{Euler}}{w_0}}{h} = 1$$

$$x_2: \quad -\varepsilon \frac{w_3 - 2w_2 - w_1}{h^2} + \frac{w_2 - w_1}{h} = 1$$

$$\Rightarrow \quad A = \begin{bmatrix} 2\varepsilon + h & -\varepsilon \\ -h - \varepsilon & 2\varepsilon + h \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} h^2 \\ h^2 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = A^{-1}\vec{b} \approx \begin{bmatrix} 0.3325 \\ 0.6376 \end{bmatrix} \qquad (\varepsilon = 0.1)$$

$\approx 1/3$

$\approx 2/3$