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Sec 11.4. Finite - Difference Methods for Nonlinear Problems.

- Newton's method for vector equations.

· $f : \mathbb{R} \to \mathbb{R}$, $\quad f(x) = 0$

$$\Rightarrow \quad x^{(k+1)} = x^{(k)} + f(x^{(k)}) / f'(x^{(k)})$$

· $\vec{f} : \mathbb{R}^m \to \mathbb{R}^m$, $\quad \vec{f}(\vec{x}) = \vec{0}$

$$\vec{f}(\vec{x}) = \begin{bmatrix} f_1(x_1, \cdots x_m) \\ \vdots \\ f_m(x_1, \cdots, x_m) \end{bmatrix}$$

$$\Rightarrow \quad \vec{x}^{(k+1)} = \vec{x}^{(k)} + J_{\vec{f}}(\vec{x}^{(k)})^{-1} \vec{f}(\vec{x}^{(k)})$$

where $\quad J_{\vec{f}}(\vec{x}) = (D\vec{f})(\vec{x})$

$\quad \hookrightarrow (J_{\vec{f}})_{ij} = \partial f_i / \partial x_j$

ex) $\quad f_1(x_1, x_2) = 0$

$\quad\quad f_2(x_1, x_2) = 0$

$$\begin{bmatrix} x_1^{(k+1)} \\ x_2^{(k+1)} \end{bmatrix} = \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \end{bmatrix} + J_{\vec{f}}(\vec{x}^{(k)})^{-1} \vec{f}(\vec{x}^{(k)})$$

$$J_{\vec{f}}(\vec{x}) = \begin{bmatrix} (\partial f_1/\partial x_1)(\vec{x}) & (\partial f_1/\partial x_2)(\vec{x}) \\ (\partial f_2/\partial x_1)(\vec{x}) & (\partial f_2/\partial x_2)(\vec{x}) \end{bmatrix}$$

$$\vec{f}(\vec{x}) = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix}$$

- Examples of FDMs.

ex) $y'' = -(y')^2 - y + \ln x$, $\quad 1 \le x \le 2$.

$y(1) = 0$, $\quad y(2) = \ln 2$, $\quad h = 0.5$.

x_0	x_1	x_2
1	1.5	2

$$x_1 : \frac{w_2 - 2w_1 + w_0}{h^2} = -\left(\frac{w_2 - w_0}{2h}\right)^2 - w_1 + \ln x_1$$

$$\Rightarrow \ln 2 - 2w_1 = -\frac{1}{4}(\ln 2)^2 - w_1 + \ln 1.5.$$

$$\Rightarrow w_1 = \frac{1}{4}(\ln 2)^2 + \ln 2 - \ln 1.5.$$

$w_0 = 0$. $\quad w_2 = \ln 2$.

ex) $y'' = 2y^3$, $\quad -1 \le x \le 0$,

$y(-1) = 1/2$, $\quad y(0) = 1/3$, $\quad h = 1/3$

x_0	x_1	x_2	x_3
-1	$-2/3$	$-1/3$	0

$$x_1 : \frac{w_2 - 2w_1 + w_0}{h^2} = 2w_1^3$$

$$x_2 : \frac{w_3 - 2w_2 + w_1}{h^2} = 2w_2^3$$

$$\Rightarrow x_1 : w_2 - 2w_1 + \frac{1}{2} = 2h^2 w_1^3$$

$$x_2 : \frac{1}{3} - 2w_2 + w_1 = 2h^2 w_2^3$$

$$\Rightarrow \quad x_1 : \quad W_2 - 2W_1 - 2h^2 W_1^3 + \frac{1}{2} = 0$$

$$x_2 : \quad \frac{1}{3} - 2W_2 - 2h^2 W_2^3 + W_1 = 0$$

$$\hookrightarrow \quad f_1(w_1, w_2) = 0$$
$$f_2(w_1, w_2) = 0$$

$$\Rightarrow \quad J_{\vec{f}}(\vec{w}) = \begin{array}{c} \begin{array}{cc} \partial f_1/\partial w_1 & \partial f_1/\partial w_2 \end{array} \\ \begin{bmatrix} -2 - 6h^2 W_1^2 & 1 \\ 1 & -2 - 6h^2 W_2^2 \end{bmatrix} \end{array}$$

$\Rightarrow$ Let $\vec{w}^{(0)} = \vec{0}$, initial guess for Newton's method. Then,

$$\vec{w}^{(1)} = \vec{w}^{(0)} + J_{\vec{f}}(\vec{w}^{(0)})^{-1} \vec{f}(\vec{w}^{(0)})$$
$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 1/2 \\ 1/3 \end{bmatrix}$$
$$= \frac{1}{3} \begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/3 \end{bmatrix}$$
$$= \frac{1}{3} \begin{bmatrix} -4/3 \\ -7/6 \end{bmatrix}$$

- Another form for computer codes

$$\vec{w}^{(k+1)} = \vec{w}^{(k)} + \vec{v}^{(k)}$$
$$\text{with} \quad J_{\vec{f}}(\vec{w}^{(k)})\, \vec{v}^{(k)} = \vec{f}(\vec{w}^{(k)})$$

- Stopping criterion

$$\| \vec{w}^{(k+1)} - \vec{w}^{(k)} \| = \| \vec{v}^{(k)} \| \leq \mathrm{TOL}.$$

Quiz 4.

1. Consider the following boundary-value problem.

$$-\varepsilon y'' + y' = 1 \quad \text{in} \quad (0, 1)$$
$$y(0) = 0, \quad y(1) = 0$$

a. Show that its exact solution is

$$y(x) = x - \frac{e^{-(1-x)/\varepsilon} - e^{-1/\varepsilon}}{1 - e^{-1/\varepsilon}}$$

b. Use the FDM with the centered difference to solve the BVP numerically. Find a proper step size to get a stable solution when 'e=0.1'. (positive)