

05/14/20

Sec 11.4. Finite-Difference Methods for Nonlinear Problems.

- Newton's method for vector equations.

$$\begin{aligned} \cdot & f : \mathbb{R} \rightarrow \mathbb{R}, \quad f(x) = 0 \\ & \Rightarrow x^{(k+1)} = x^{(k)} + f(x^{(k)}) / f'(x^{(k)}) \end{aligned}$$

$$\begin{aligned} \vec{f}(\vec{x}) = & \begin{bmatrix} f_1(x_1, \dots, x_m) \\ \vdots \\ f_m(x_1, \dots, x_m) \end{bmatrix} \quad \cdot \quad \vec{f} : \mathbb{R}^m \rightarrow \mathbb{R}^m, \quad \vec{f}(\vec{x}) = \vec{0} \\ & \Rightarrow \vec{x}^{(k+1)} = \vec{x}^{(k)} + J_{\vec{f}}(\vec{x}^{(k)})^{-1} \vec{f}(\vec{x}^{(k)}) \end{aligned}$$

where $J_{\vec{f}}(\vec{x}) = (D\vec{f})(\vec{x})$
 $\hookrightarrow (J_{\vec{f}})_{ij} = \partial f_i / \partial x_j$

ex) $f_1(x_1, x_2) = 0$
 $f_2(x_1, x_2) = 0$

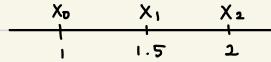
$$\begin{bmatrix} x_1^{(k+1)} \\ x_2^{(k+1)} \end{bmatrix} = \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \end{bmatrix} + J_{\vec{f}}(\vec{x}^{(k)})^{-1} \vec{f}(\vec{x}^{(k)})$$

$$J_{\vec{f}}(\vec{x}) = \begin{bmatrix} (\partial f_1 / \partial x_1)(\vec{x}) & (\partial f_1 / \partial x_2)(\vec{x}) \\ (\partial f_2 / \partial x_1)(\vec{x}) & (\partial f_2 / \partial x_2)(\vec{x}) \end{bmatrix}$$

$$\vec{f}(\vec{x}) = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix}$$

- Examples of FDMs .

ex) $y'' = -(y')^2 - y + \ln x , \quad 1 \leq x \leq 2 ,$
 $y(1) = 0 , \quad y(2) = \ln 2 , \quad h = 0.5 .$



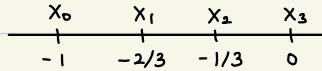
$$x_1 : \frac{w_2 - 2w_1 + w_0}{h^2} = - \left(\frac{w_2 - w_0}{2h} \right)^2 - w_1 + \ln x_1$$

$$\Rightarrow \ln 2 - 2w_1 = - \frac{1}{4} (\ln 2)^2 - w_1 + \ln 1.5 .$$

$$\Rightarrow w_1 = \frac{1}{4} (\ln 2)^2 + \ln 2 - \ln 1.5 .$$

$$w_0 = 0 , \quad w_2 = \ln 2 .$$

ex) $y'' = 2y^3 , \quad -1 \leq x \leq 0 ,$
 $y(-1) = 1/2 , \quad y(0) = 1/3 , \quad h = 1/3$



$$x_1 : \frac{w_2 - 2w_1 + w_0}{h^2} = 2w_1^3$$

$$x_2 : \frac{w_3 - 2w_2 + w_1}{h^2} = 2w_2^3$$

$$\Rightarrow x_1 : w_2 - 2w_1 + \frac{1}{2} = 2h^2 w_1^3$$

$$x_2 : \frac{1}{3} - 2w_2 + w_1 = 2h^2 w_2^3$$

$$\Rightarrow x_1 : w_2 - 2w_1 - 2h^2 w_1^3 + \frac{1}{2} = 0$$

$$x_2 : \frac{1}{3} - 2w_2 - 2h^2 w_2^3 + w_1 = 0$$

$$\hookrightarrow f_1(w_1, w_2) = 0$$

$$f_2(w_1, w_2) = 0$$

$$\Rightarrow J_{\vec{f}}(\vec{w}) = \begin{bmatrix} \frac{\partial f_1}{\partial w_1} & \frac{\partial f_1}{\partial w_2} \\ \frac{\partial f_2}{\partial w_1} & \frac{\partial f_2}{\partial w_2} \end{bmatrix}$$

\Rightarrow Let $\vec{w}^{(0)} = \vec{0}$, initial guess for Newton's method. Then,

$$\begin{aligned} \vec{w}^{(1)} &= \vec{w}^{(0)} + J_{\vec{f}}(\vec{w}^{(0)})^{-1} \vec{f}(\vec{w}^{(0)}) \\ &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 1/2 \\ 1/3 \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/3 \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} -4/3 \\ -7/6 \end{bmatrix} \end{aligned}$$

- Another form for computer codes

$$\vec{w}^{(k+1)} = \vec{w}^{(k)} + \vec{v}^{(k)}$$

$$\text{with } J_{\vec{f}}(\vec{w}^{(k)}) \vec{v}^{(k)} = \vec{f}(\vec{w}^{(k)})$$

- Stopping criterion

$$\|\vec{w}^{(k+1)} - \vec{w}^{(k)}\| = \|\vec{v}^{(k)}\| \leq TOL.$$

Quiz 4.

1. Consider the following boundary-value problem.

$$-\varepsilon y'' + y' = 1 \quad \text{in } (0, 1)$$
$$y(0) = 0, \quad y(1) = 0$$

- a. Show that its exact solution is

$$y(x) = x - \frac{e^{-(1-x)/\varepsilon} - e^{-1/\varepsilon}}{1 - e^{-1/\varepsilon}}$$

- b. Use the FDM with the centered difference to solve the BVP numerically. Find a proper step size to get a stable solution when ' $\varepsilon=0.1$ '.
(positive)