05/19/20

Sec 11.5. The Rayleigh – Ritz Method

$$-\left(p(x)y'(x)\right)' + q(x)y(x) = f(x)$$

for $0 \le x \le 1$, $y(0) = y(1) = 0$.

$p(x) > 0$, $q(x) \ge 0$

- Variational Problems (weak formulation)

Find $y \in C_0^2[0,1]$ such that

$$\int_0^1 p(x)y'(x)u'(x) + q(x)y(x)u(x)\,dx$$

$$= \int_0^1 f(x)u(x)\,dx$$

for all $u \in C_0^2[0,1]$.

(from integration by parts)

$$\Updownarrow$$

$y \in C_0^2[0,1]$ minimizes

$$I[u] = \int_0^1 p[u']^2 + q[u]^2 - 2fu\,dx.$$

$\Rightarrow$ It is hard to find such $y \in C_0^2[0,1]$ because the space $C_0^2[0,1]$ is infinite dimensional.

- Rayleigh – Ritz method

finite-dimensional

Find $y_h \in V_h[0,1]$ such that

test function.

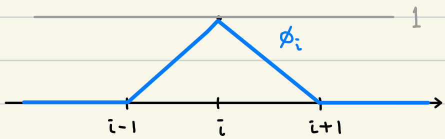
$$\int_0^1 p y_h' u_h' + q y_h u_h \, dx = \int_0^1 f u_h \, dx$$

for all $u_h \in V_h[0,1]$.

(from the variational problem)

- Piecewise linear basis

$$u_h = \sum_i c_i \phi_i$$

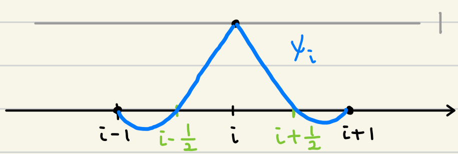


$\dim(V_h) = \#$ of mesh points -2

boundary condition.

- Piecewise quadratic basis

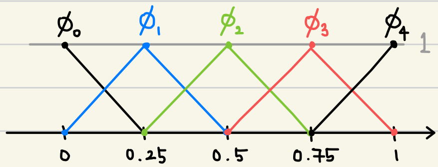
$$u_h = \sum_i d_i \psi_i$$



$\dim(V_h) = 2 \times (\#$ of mesh points$) - 1$

ex) $-y'' = 1, \quad 0 \le x \le 1,$

$y(0) = y(1) = 0, \quad h = 0.25$



$$y_h(x) = \sum_{i=0}^{4} c_i \phi_i(x)$$

$(c_0, c_1, \cdots, c_4)$

Find $y_h \in V_h$ such that test function ⇒ basis function.

$$\int_0^1 y_h' \phi_j' \, dx = \int_0^1 \phi_j \, dx$$

for all $0 \le j \le 4$.

(because ϕ_j are basis functions with $\dim(V_h) = 5$)

$y(0) = y(1) = 0 \quad \Rightarrow \quad c_0 = c_4 = 0$

$(y(0) = \alpha, \ y(1) = \beta \quad \Rightarrow \quad c_0 = \alpha, \ c_4 = \beta)$

Unknowns : c_1, c_2, c_3

Equations : $\int_0^1 y_h' \phi_j' \, dx = \int_0^1 \phi_j \, dx$

$1 \le j \le 3$.

$$\Rightarrow \int_0^1 \left(\sum_{i=1}^{3} c_i \phi_i \right)' \phi_j' \, dx = \int_0^1 \phi_j \, dx$$

$$\Rightarrow \sum_{i=1}^{3} c_i \underbrace{\int_0^1 \phi_i' \phi_j' \, dx}_{a(\phi_i, \phi_j)} = \underbrace{\int_0^1 \phi_j \, dx}_{b(\phi_j)}$$

$$\Rightarrow \begin{bmatrix} a(\phi_1,\phi_1) & a(\phi_2,\phi_1) & a(\phi_3,\phi_1) \\ a(\phi_1,\phi_2) & a(\phi_2,\phi_2) & a(\phi_3,\phi_2) \\ a(\phi_1,\phi_3) & a(\phi_2,\phi_3) & a(\phi_3,\phi_3) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} b(\phi_1) \\ b(\phi_2) \\ b(\phi_3) \end{bmatrix}$$

$$(A\vec{c} = \vec{b})$$

$$\hookrightarrow \ a(\phi_1,\phi_1) = \int_0^{0.5} \phi_1' \phi_1' \, dx$$

$$= (0.25)[4^2 + (-4)^2] = 8$$

$$a(\phi_2,\phi_1) =$$

$$a(\phi_3,\phi_1) = 0$$

$\Rightarrow \quad A\vec{c} = \vec{b}$, where

$$A = \begin{bmatrix} 8 & -4 & 0 \\ -4 & 8 & -4 \\ 0 & -4 & 8 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 0.25 \\ 0.25 \\ 0.25 \end{bmatrix}$$

because $b(\phi_j) = \int_0^1 \phi_j \, dx = \frac{1}{2} \times 1 \times 0.5$.